Construction/Destruction Monte Carlo to Estimate the Number of Path-sets for the Reliability Measure of a Network with Diameter Constraints

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Abstract—Consider a network $G = (V, E, T)$ composed of a set $V$ of vertices, a set $E$ of edges, and a set $T$ of vertices called the terminal-set. In this paper we show how the Construction/Destruction (C/D) Monte Carlo simulation methods can be applied to estimate the number of topological structures of a graph $G$, called path-sets (and their combinatorial dual cut-sets), which are subgraphs of $G$ where the maximum distance between the vertices of $T$ (i.e., the $T$-diameter) is less than or equal to a given diameter bound $d$. The number of path-sets (or cut-sets) are required to calculate the Diameter-constrained reliability of a network, and since determining the number of such structures is an intractable problem, we perform computational tests on different graphs to demonstrate the accuracy of the C/D techniques.

Keywords: Graph theory, network reliability, monte carlo simulation methods, computational complexity.

1. Introduction

The system under study is a communication network represented by an undirected graph $G = (V, E, T)$, where $V$ and $E$ are the sets of vertices and edges of $G$, and $T \subseteq V$ is the so-called terminal set. Under the assumption that edges fail independently with known probabilities (vertices are always reliable), the classical reliability (CR), $R_T(G)$, is the probability that after deletion of the failed edges, there exists a path connecting each pair $u, v \in T$ of terminal vertices.

For a fixed size integer bound $d$, the $T$-terminal Diameter-constrained network reliability (DCR), denoted as $R_T(G, d)$, introduced by Petingi and Rodriguez in 2001 [1], is the probability that there exists a path of length at most $d$ between each pair of terminal vertices $u$ and $v$, or equivalently, the maximum distance (i.e., $T$-diameter) between terminal vertices is at most $d$. If the vertices of $T$ belong to different connected components then we let $T$-diameter $= \infty$.

In the classical reliability the distances between vertices are not taken into account. The Diameter-constrained reliability can be useful in different contexts in which the distances between vertices play a major role for assessing performance objectives of a network. For example, this measure gives an indicator of the suitability of an existing network topology to support good quality voice over IP applications between a pair of terminals. Another example of its applicability is in the case of packet-oriented networks where links may fail and there is a "time-to-live" limit, specified by number of hops (i.e., the diameter bound $d$) that can be traversed by any given packet (for instance, IPv6 packets include a hop limit field [2]).

The DCR is a generalization of the classical reliability as the maximum length of a path in a network on $n$ vertices is of at most $n - 1$ edges, then $R_T(G, d) = R_T(G)$, whenever $d = n - 1$. As computation of the classical reliability, for arbitrary terminal-set $T$, is an NP-hard problem [3], then determination of the DCR is a NP-hard problem as well. For fixed number of terminal vertices $T$, and for fixed diameter bound $d$, it was shown in [5] that computing $R_T(G, d)$ is also an NP-hard problem. To address the intractability, we study Monte Carlo (MC) Construction/Destruction methods (CDM) [4], [5], [6], [7], that were originally conceived under the frame of the classical reliability. CDM provides estimates for some topological structures (path- and cut-sets) and as a by-product, estimates for the reliability measure.

Given a graph $G = (V, E, T)$, a path-set of $G$ is a set of $r$ edges that induces a subgraph of $G$ in which all the terminal vertices are contained within a connected component; let $S_r$ be the number of path-sets of $G$ of size $r$. Similarly, a cut-set is a set of $r$ edges that if deleted from $G$ will disconnect at least a pair of terminal vertices of $T$; let $C_r$ be the number of cut-sets of $G$ of size $r$. The CDM provide estimates for $S_r$ and $C_r$, and therefore for the classical reliability as it will be shown in the next section. The concept of path-set can be then extended for any given integer $d$ (d-path-set), that is, all terminal vertices belong to a connected component whose $T$-diameter is at most $d$. Similarly a $d$-cut-set is a set of edges whose removal from $G$ will result in a graph whose $T$-diameter is greater than $d$. In this paper we extend the Monte Carlo CDM to estimate the number of $d$-path-sets and $d$-cut-sets of different edge sizes, and therefore also providing estimates for the DCR as a by-product. This is an important contribution since depending on the terminal-set $T$, computation of the number of cut-sets and path-sets for specific sizes, may be NP-hard problems (see for example [8], page 31). In this work, we particularly concentrate on the Construction (CM) Monte Carlo method.

The paper is structured as follows. In the next section we introduce relevant background and present the MC Construction/Destruction techniques and an algorithm based on

2. Construction-Destruction Methods to Estimate the Number of Path- and Cut-sets and Evaluation of the DCR

2.1 Construction Model

Let \( q(e) = 1 - p(e) \) be the probability of failure of the edge \( e \) (i.e., \( p(e) \) is the edge-reliability of \( e \)).

Given a diameter bound \( d \) and under the assumption that the edges fail independently with equal probability \( q = 1 - p \), the Diameter-constrained reliability can be computed as [9]:

\[
R_T(G, d, p) = \sum_{i=1}^{m-|E|} S_i^d p^i (1 - p)^{m-i},
\]

(1)

where \( S_i^d \) is the number of \( d \)-path-sets of size \( i \), and \( l_d \) is the length of a minimum size \( d \)-path-set (min-path).

As the problem of calculating the reliability is NP-hard, to compute at least one of \( S_i^d \)'s must be then \#P-complete. The Construction method works as follows: generate a random permutation of the edges \( \pi = \{i_1, i_2, ..., i_m\} \), put all edges down and turn one edge at the time up, from left to right, until the graph becomes \( UP \).

In the DCR setting, \( UP \) state is defined as the presence of a path of length \( L \leq d \) between each pair of terminal vertices. This condition can be tested by application of Floyd’s algorithm [10] of complexity \( O(n^3) \) for a graph on \( n \) vertices.

Assuming that all \( \pi \) are equal-probable, denoted by \( g_i \), the probability that the transition \( DOWN \rightarrow UP \) takes place on the \( i \)-th step of the Construction.

The collection \( g = \{g_1, ..., g_m\} \) is called the C-spectrum ([6], page 142). In further exposition, the main role is played by the so-called Cumulative C-spectrum \( v(k) \), \( k = 1, 2, ..., m \) defined as \( v(k) = \sum_{i=1}^{k} g_i, k = 1, ..., m \). \( \{v(k)\} \) are system structural invariants and have the outstanding property that they allow counting the system path-sets, by the virtue of the following. Let \( S_k^d \) be the number of \( d \)-path-sets of size \( k \), i.e. having \( k \) edges up and remaining \( m - k \) down. Then

\[
S_k^d = v(k) \frac{m!}{k!(m-k)!}.
\]

(2)

So, if we simulate \( M \) random permutations and count the number \( M_k \) of them with the \( DOWN \rightarrow UP \) transition on the \( i \)-th step of C-process, then \( \sum_{j=1}^{k} M_j/M \) will be an unbiased estimate of \( v(k) \).

The proof of the theorem is combinatorial and it is a dual variation of the similar proof given in page 16 of [7] for the so-called Destruction method. The process starts when all components of the random permutations are set to up and sequentially turned down until the system enters the \( DOWN \) state. The proof therefore remains the same by interchanging \( UP \) and \( DOWN \). The number of \( d \)-path-sets and \( d \)-cut-sets are related by the following equality

\[
C_j^d + S_{m-j}^d = \binom{m}{j},
\]

(3)

Remark 1: The Destruction process produces, similar to the Construction process, the so-called D-spectrum.

\[ f = \{f_1, f_2, ..., f_n\}, \text{ where } f_i \text{ is the probability that the transition } UP \rightarrow DOWN \text{ takes place on the } i \text{-th step of the process.} \]

As in the Construction process, we define the Cumulative D-spectrum as \( b(k), k = 1, 2, ..., m \), where \( b(k) = \sum_{i=1}^{k} f_i, k = 1, ..., m \), where both Cumulative spectra are connected by \( b(j) = 1 - v(m - j) \). Consider the index of the first positive quantity in \( f_j \): \( f_1 = 0, ..., f_{j-1} = 0, f_j > 0 \). Then obviously \( j \) is the size of the minimum cut-set (i.e., \( d \)-cut), and \( C^d_j = f_j m!/(j!(m-j)!) \) is the number of such sets. From here follows the Burtin-Pittel approximation (see [5], page 36) for network \( DOWN \) probability, with \( q \rightarrow 0: P(DOWN) = C^d_j \cdot q^j (1 + o(1)). \)

The relative-error of a non-negative r.v. \( X \) is defined as \( re[X] = \sigma_X/E[X] \), where \( \sigma_X \) and \( E[X] \) are the standard-deviation and expected-value of \( X \), respectively.

As \( v(k) = \text{Prob} \{ \text{transition from a } DOWN \text{ to a } UP \text{ state takes place before or at the } k \text{-th step of the process.} \} \) the relative-error for the unbiased estimator of \( v(k) \), \( \hat{v}(k) \) (see [6], page 10), is

\[
\hat{re}[v(k)] = \sqrt{1 - v(k)}/\sqrt{M \cdot v(k)}.
\]

Moreover it can be easily shown that for fixed \( M \), \( \hat{re}[v(k)] \) is monotonically non-increasing as \( k \) increases.

Here \( M \) is the number of trials (random permutations of the edges).

2.2 Construction Algorithm

In this section we present a serial MC implementation of the CM method to calculate the number of \( d \)-path-sets of different sizes as discussed in the previous section:

1) \text{Input: Probabilistic graph } G = (V,E,T), \text{ where } V \text{ and } E \text{ are the sets of vertices and edges of } G, \text{ and } T \subseteq V \text{ is the terminal set, and a diameter bound } d.

2) \text{Let } N_i = 0, 1 \leq i \leq m \text{ (Where } N_i \text{ is the number of permutations with critical number } i).\n
3) \text{Repeat } M \text{ times, where } M \text{ is the number of trials:}
   a) Generate a random permutation of the edges, \( \pi = \{e_1, e_2, ..., e_m\} \).
   b) Starting with the empty graph, add edges from \( \pi \) from left to right until the T-diameter \( \leq d \) (Floyd’s algorithm).
   c) If \( e_k \) is the first edge for which the system goes \( UP \) then \( k \) is the critical number for \( \pi \), then \( N_k = N_k + 1 \).
4) End Repeat.
5) Let the C-Spectrum be defined as the collection of \( g = \{ g_1, g_2, \ldots, g_m \} \) where \( g_s \) is the probability that the critical number is \( s \), meaning \( g_s = \frac{N}{M^s} \).
6) Let the Cumulative C-spectrum be defined as the collection of \( v = \{ v(1), v(2), \ldots, v(m) \} \) where \( v(s) \) is the probability that the critical number is \( s \) or less, meaning \( v(s) = \sum_{i=1}^{s} g_i \).
7) Let the number of \( d \)-path-sets, \( S^d_s \), with \( s \) edges be defined as, \( S^d_s = v(r) \frac{m^r}{r!(m-r)!} \), \( 1 \leq r \leq m \).

3. Computational Tests

The Construction algorithm discussed in the previous section was implemented in C++ and run on a HP 1.6 Ghz processor. The complexity of the algorithm is \( O(M \cdot \log_2 m \cdot n^3) \) where \( M \), \( n \) and \( m \) are the number of trials, and vertices and edges respectively of the tested topologies; Floyd’s algorithm [10] was applied to determine if the Min-cuts (see Fig. 1), composed of 16 vertices and 32 edges, and 20 vertices and 30 edges, respectively. The maximal distance \( d \) and for diameter bounds were sampled. This is consistent with the monotonically non-increasing relative-error of \( v(k) \) (as therefore the relative-error of \( S^d_s \) stated in Remark 2, at the end of Section 2.1, as \( k \to m \). As a consequence of Equation 1, the Diameter-constrained reliability can be also estimated as a function of path-sets (see Table 1), but determination of the accuracy of the results yielded is not within the scope of this work.

Table 1

<table>
<thead>
<tr>
<th>Topology</th>
<th>7</th>
<th>Exact Reliability</th>
<th>Estim. Reliability</th>
<th>% error</th>
</tr>
</thead>
<tbody>
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<td>Dodec. (0,19)</td>
<td></td>
<td>0.9967389962</td>
<td>0.99673967938</td>
<td>6.85E-05</td>
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<tr>
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<td></td>
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<td>0.999979411258</td>
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</tbody>
</table>

In the next section we show how the size of a minimum \( d \)-cut can be estimated using the CM technique; for the case \( T = \{ s, t \} \) and \( d \) is of unconstrained length, the size of a \( d \)-min-cut is computed by means of the Max-Flow Min-cut Theorem [11], however the problem is NP-hard whenever considering paths of bounded-length \( d \), for \( d > 4 \) [12].

4. Estimating the Number and Size of Min-cuts

In this section we focus on establishing the accuracy of the Monte Carlo CM for estimating the size of a \( d \)-min-cut of a graph; this problem is NP-hard when considering paths of bounded-length \( d \), for \( d > 4 \) [12], and when the terminal-set is \( T = \{ s, t \} \) (and therefore for arbitrary number of terminal vertices). Consider the Cumulative D-spectrum discussed in Remark 1 of Section 2.1, where the D-spectrum \( f = \{ f_1, f_2, \ldots, f_n \} \). The index \( j \) of the first \( f_j \): \( f_1 = 0, \ldots, f_{j-1} = 0, f_j > 0 \), is the size of the minimum cut-set (i.e., \( d \)-min-cut). By connecting the Cumulative D-spectrum with the Cumulative C-spectrum (see Remark 1), \( f_j = b(j) = 1 - v(m - j) \), where \( m \) is the number of edges of \( G \). Thus in terms of the Cumulative C-spectrum, the size of a \( d \)-min-cut is the first \( j \) for which \( v(m - j) < 1 \).

Consider the Cumulative C-spectrum of the Dodecahedron and 4-dimensional Hypercube (see Fig. 1) shown in Table 3 as calculated by the CM algorithm. For the Dodecahedron graph, and for diameter bounds \( d = 7 \) and \( d = 15 \), the first \( v(m - j) < 1 \) (\( v(k) \) was rounded-up to 1.0 when \( 1 - v(k) < 10^{-15} \)) is when \( m - j = 27 \), therefore \( j = 3 \) (the size of the \( d \)-min-cut) as \( m = 30 \). Similarly for the 4-dimensional Hypercube graph, and for \( d = 7 \) and \( d = 15 \), the first \( v(m - j) < 1 \) is when \( m - j = 28 \), therefore \( j = 4 \) as \( m = 32 \).

![Fig. 1](image_url)

The number of \( d \)-min-cuts can be estimated by application of Burtn-Pittel approximation formula (Remark 1) originally introduced within the frame of the classical reliability and extended here for the DCR case. As an example let’s consider the Dodecahedron graph depicted in Figure 1 (terminal
Table 2
COMPARISON OF THE ESTIMATED NUMBER OF d-PATH-SETS YIELDED BY THE CONSTRUCTION METHOD WITH THE EXACT NUMBER OF d-PATH-SETS OF DIFFERENT EDGE-SIZES (DETERMINED BY A BACKTRACKING PROCEDURE OF EXPONENTIAL COMPLEXITY) FOR THE 4-DIMENSIONAL HYPERCUBE AND THE DODECAHEDRON TOPOLOGIES.

<table>
<thead>
<tr>
<th>G</th>
<th>d</th>
<th>1-3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>18</th>
<th>22</th>
<th>29</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-dim cube</td>
<td>Exact</td>
<td>7</td>
<td>0</td>
<td>24</td>
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<tr>
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<td>0</td>
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<td>0.0841</td>
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<tr>
<td>4-dim cube</td>
<td>Exact</td>
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<td>0</td>
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<td>672</td>
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<td></td>
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<td>24</td>
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<td>0</td>
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<td>4947897</td>
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<td>Estimated</td>
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<td>0</td>
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<td>0</td>
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</tr>
<tr>
<td>Dode</td>
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<td>19</td>
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<td>0</td>
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<td>0</td>
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<td>0.188</td>
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<td>0.0411</td>
</tr>
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Table 3
CUMULATIVE C-SPECTRUM v(i), 1 ≤ i ≤ m, FOR THE DODECAHEDRON AND 4-DIM CUBE TOPOLOGIES, FOR DIAMETER BOUNDS d = 7 and d = 15.

<table>
<thead>
<tr>
<th>G</th>
<th>d</th>
<th>1-3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>18</th>
<th>22</th>
<th>29</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-dim cube</td>
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<tr>
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</tr>
</tbody>
</table>

By Burtin-Pittel approximation, \( P_1(\text{down})/P_2(\text{down}) = (q_1/q_2)^c \), where \( c \) is the number of d-min-cuts (i.e., \( C^G_f \)). Therefore \( c = \log(P_1(\text{down})/P_2(\text{down}))/\log(q_1/q_2) = 3.042766 \), thus by rounding we then get \( c = 3 \).

5. Conclusions

Construction/Destruction Models have shown to be useful MC techniques to estimate the classical reliability measure and to count combinatorial structures of graphs, many of them shown to be computationally intractable problems. In this paper we validate their use by presenting extensions to compute the Diameter-constrained reliability and to count related topological structures (i.e., d-path-sets and d-cut-sets) given a fixed diameter bound \( d \). To achieve such objective we’ve presented a modified version of the Construction simulation method to evaluate the DCR in terms of the number of d-path-sets with given number of edges and of their dual topological structures (cut-sets). We’ve also showed how these techniques can yield good estimates for computationally expensive combinatorial problems as for example to calculate the number and size of minimum \( d \)-cuts, for fixed restricted diameter bounds \( d > 4 \).

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