Finite Difference Method for Non-smooth Beam Bending

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Abstract—The finite difference method (FDM) is widely used numerical method for numerical computation of different physical problems. The method is very useful for time-dependent problems. It has also wide implementation in linear structural analysis before finite element method. The sequence of finite difference method solutions with increasing the number of discrete points converge to analytical solutions. It converges rapidly for smooth problems. Different structural engineering problems with discontinuity in derivatives are considered. The simple structural engineering problem with concentrated force has discontinuity in third derivative. The problem is how to model that kind of loading for finite difference method. New approach is explained. The concentrated force is not taken as loading. It is taken as the difference of third derivative at loading point. Discontinuity in geometrical properties under standard loading leads to discontinuity in second derivative. The procedures for numerical evaluation of both problems are described. The presented numerical example shows that the sequence of numerical results for beam under concentrated forces converges with quadratic order of convergence to analytical solution. The numerical results for beams with geometrical discontinuity converges to analytical solution.

Keywords: finite difference method, beam bending, concentrated force, geometrical discontinuity

1. Introduction

The finite difference method (FDM) is widely used for numerical computation of different physical problems. The method is based on the discretization of the domain on the points. The simple structural engineering problems with different kind of discontinuities are considered.

The structural engineering problem of beam bending with concentrated force has discontinuity in third derivative at the point under concentrated force. The beam bending problem is fourth-order boundary value problem. Finite difference scheme is calculated and applied for fourth derivative of displacement function. Concentrated force, as mathematical function, is discontinuous function over beam as domain. The function is equal to force value at the point of loading but zero at all other beam points. Applying the standard FDM scheme with loading on the right side of equation doesn’t lead to solution that converges. Even the measure unit doesn’t fit. The concentrated force is taken into account according its physical influence on the beam. The concentrated force gives discontinuity in transverse force at the loading point. It means that the third derivative of displacement function has discontinuity at the loading point. The concentrated force is not taken as usual as loading. It is taken as discontinuity in third derivative at loading point. The third derivative of displacement function very close left to loading point is not same as the third derivative of displacement function very close right to loading point. The difference between transverse forces is equal to given concentrated force. It has been shown how to make numerical model to get sequence of numerical results that converges to analytical solution. The procedure is shown on simple numerical example. The third derivative is approximated only with function values at the points on the same side of the loading point. The sequence of numerical results converges to analytical result with quadratic order of convergence.

The structural engineering problem of beam bending with discontinuity in geometrical properties of the beam (discontinuity in the beam height followed by discontinuity in moment of inertia and discontinuity in flexural rigidity) leads to solution with discontinuity in second derivative at the point with discontinuity in geometrical properties. Standard finite difference scheme can not be applied in the points with geometrical discontinuity because the flexural rigidity is not defined at the point of discontinuity. The flexural rigidity is taken as weighted coefficient of distributed load. Distributed load, divided with flexural rigidity over beam, has now discontinuity at the point with discontinuity in geometrical properties. The loading function at that point is approximated with mid-value of loading function from both sides. It has been shown how to make numerical model to get the sequence of numerical results that converges to analytical solution. The procedure is shown on simple numerical example. The sequence of numerical results converges to analytical solution.

2. Finite difference for beam bending problem

The finite difference method approximate the derivatives with function values at discrete points. Let \( w(x) \) represent a function of one variable. The function is assumed to be enough smooth. It means that it has all needed derivatives over an interval containing a particular point \( x_i \). The needed \( n \)-th derivative at point \( x_i \) is approximate as linear combination of \( n + 1 \) points of the domain. The error of approximation is of order equal to \( n + 1 \). The different choice of points taken for linear combination leads.
The governing differential equation for the beam bending problem is

$$EI w'''' = q(x).$$

If we use FDM scheme for fourth derivative in first two or last two points of the beam, we have unknown function values outside the beam domain. We can avoid this unknown function values in FDM scheme. With explicit boundary conditions in FDM scheme, we can avoid this unknown function values outside the beam domain.

For simply supported edge, moment is equal to zero. The expression for transverse force function is

$$T_1(x) = -EI w''''(x).$$

For free edge, we know that moment and transverse force at free edge are equal to zero. For moment value at free edge we have

$$M_1 = -EI w''' = 0.$$
3. Modelling the concentrated force in FDM scheme

The beam under concentrated force at any point doesn’t have third derivative in that observed point. The value of transverse force close to point on the left is different to value close to the point from right. The difference between transverse forces is equal to given concentrated force at that point. It is obvious that there is also no fourth derivative at that point. The question is how to involve concentrated load in FDM scheme to get valuable solution that converges with increasing number of points. The problem has to be viewed in its physical meaning. We have to ask ourselves what is really happened in that point. We have discontinuity in the third derivative of displacement function. That means, we do not have third derivative very close left and very close right to the point under concentrated force. The third derivative is not defined at the point of concentrated force. The approximation of the loading function in point of the discontinuity could be taken as mid-value of loading from both sides as

\[
\bar{q}(\alpha L) = \frac{q}{nEI} + \frac{q}{mEI} = \frac{q}{EI} \frac{n + m}{2m}. \tag{19}
\]

5. Numerical examples

Example 1: Let we apply given procedure to simply supported beam of length \(L\) and flexural rigidity \(EI\) under concentrated forces. The example with concentrated force at midpoint \(x_K = L/2\) of simply supported beam is described in [5]. In this example, under consideration is simply supported beam with concentrated forces at quarter of the beam length, Figure 3.

![Beam with concentrated force](image)

Let we apply given procedure to simply supported beam of length \(L\) and flexural rigidity \(EI\) under concentrated forces. The example with concentrated force at midpoint \(x_K = L/2\) of simply supported beam is described in [5]. In this example, under consideration is simply supported beam with concentrated forces at quarter of the beam length, Figure 3.

\[
K[L/4] = KL^3/512EI \begin{bmatrix}
    w_1 \\
    w_2 \\
    w_3 \\
    w_4 \\
    w_5 \\
    w_6 \\
    w_7
\end{bmatrix} = \begin{bmatrix}
    0 \\
    1 \\
    0 \\
    1 \\
    0 \\
    1 \\
    0
\end{bmatrix}, \tag{20}
\]

with stiffness matrix calculated from FDM scheme,

\[
K = \begin{bmatrix}
    5 & -4 & 1 & 0 & 0 & 0 & 0 \\
    2 & -3 & 3 & -3 & 1 & 0 & 0 \\
    1 & -4 & 6 & -4 & 1 & 0 & 0 \\
    1 & -3 & 3 & -2 & 3 & -3 & 1 \\
    0 & 0 & 1 & -4 & 6 & -4 & 1 \\
    0 & 0 & 1 & -3 & 3 & -3 & 2 \\
    0 & 0 & 0 & 0 & 1 & -4 & 5
\end{bmatrix}. \tag{21}
\]
The rows of stiffness matrix for points under concentrated forces are expressed by using equation (16) for discontinuity in third derivative of displacement function, for discontinuity in transverse force. All other rows are same as in standard procedure with approximation of the fourth derivative of displacement function. After solving the system of equation, we get displacement at midpoint, \( w \left( \frac{L}{2} \right) = \frac{KL^3}{8EI} \). Analytical solution is known, \( w \left( \frac{L}{2} \right) = \frac{K L^3}{36EI} \). With discretization over more points, we get sequence of numerical solutions that converges to analytical solution, Table 1.

### Table 1: Displacement at mid-point.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( w \left( \frac{L}{2} \right) / \left( \frac{K L^3}{8EI} \right) )</th>
<th>error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>11/512</td>
<td>3.125%</td>
</tr>
<tr>
<td>17</td>
<td>43/2048</td>
<td>0.78125%</td>
</tr>
<tr>
<td>33</td>
<td>191/8192</td>
<td>0.1953125%</td>
</tr>
<tr>
<td>exact</td>
<td>1/48</td>
<td></td>
</tr>
</tbody>
</table>

It is obvious that we get quadratic order of convergence. It means that double more points (double less mesh size) leads to quadruple better numerical solutions. The error for mesh with double more points is four times less than the error with former number of points with double larger mesh size.

**Example 2:** Let we apply given procedure to simply supported beam of length \( L \) under uniformly distributed load with flexural rigidity equal to \( 2EI \) on the first half of the beam length and flexural rigidity equal to \( EI \) on the second half of beam length, Figure 4.

![Fig. 4: Simply supported beam with discontinuous flexural rigidity](image)

Let we show the presented procedure with discretization of the beam on 9 points, \( x_i = \frac{L}{9}, i = 0, \ldots, 8 \). After applying the finite difference scheme at any point, equation (4), boundary conditions \( w_0 = w_9 = 0 \) and approximation for the point with discontinuity in flexural rigidity, as introduced in equation (19), we get the system of equation,

\[
K = \begin{bmatrix}
5 & -4 & 1 & 0 & 0 & 0 & 0 \\
-4 & 6 & -4 & 1 & 0 & 0 & 0 \\
1 & -4 & 6 & -4 & 1 & 0 & 0 \\
0 & 1 & -4 & 6 & -4 & 1 & 0 \\
0 & 0 & 1 & -4 & 6 & -4 & 1 \\
0 & 0 & 0 & 1 & -4 & 5 & | 0 \\
0 & 0 & 0 & 0 & 1 &-4 & 5
\end{bmatrix}, \quad (23)
\]

with standard stiffness matrix calculated from FDM scheme,

\[
K = \frac{qL^4}{4096EI}. \quad \text{Analytical solution is known, } w \left( \frac{L}{2} \right) = \frac{qL^4}{4096EI}. \quad \text{With discretization over more points, we get sequence of numerical solutions that converges to analytical solution, Table 2.}

### Table 2: Displacement at mid-point.

<table>
<thead>
<tr>
<th>( m )</th>
<th>( w \left( \frac{L}{2} \right) / \left( \frac{K L^3}{8EI} \right) )</th>
<th>error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>81/8192</td>
<td>1.25%</td>
</tr>
<tr>
<td>17</td>
<td>321/32768</td>
<td>0.3125%</td>
</tr>
<tr>
<td>33</td>
<td>1281/131072</td>
<td>0.078125%</td>
</tr>
<tr>
<td>exact</td>
<td>5/512</td>
<td></td>
</tr>
</tbody>
</table>

It is again obvious that we get quadratic order of convergence for the sequence of numerical solutions calculated according the presented procedure for the beam with discontinuity in flexural rigidity.

6. Conclusions

The concept for modeling the different discontinuities in the beam bending problems is explained for two different types of discontinuities. First, it was given an algorithm for the modelling of concentrated load as the discontinuity in transverse force, as the discontinuity in third derivative of displacement function. Second, it was described how to model discontinuity in flexural rigidity of the beam, discontinuity in second derivative of displacement function.

The proposed procedures for both problems are applied on numerical examples taken from standard beam bending...
problems in linear structural analysis. Numerical implementation of given procedures is described on FDM scheme with 9 discretization points. Numerical solutions are also calculated with different number of discretization points to find the order of convergence of given algorithms. The both numerical implementations for presented algorithms show expected quadratic convergence of the sequences of numerical solutions. It has been shown that proposed numerical procedures resulted with accurate and efficient numerical solutions with second order of convergence to analytical solution.

References


