Lax Pair of Discrete Nahm Equations and its Application

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Abstract—*A q-analog of Nahm equations is described in the paper by Kamata and Nakamula. In this paper, we propose other discrete Nahm equations different from that in the paper authored by Kamata and Nakamula. Our discrete analog of Nahm equations includes ordinary difference equations and has a Lax representation, although a Lax pair of the q-analog Nahm equations is not found. For numerical simulations, ordinary difference equations are more suitable than q-analog difference equations. As a reduction from our discrete analog of Nahm equations, we can extend the discrete Euler top, which we introduced in 2000, to a matrix version of the discrete Euler top using the key result that the continuous Euler top can be derived from the continuous Nahm equations. We define the equations as a discrete matrix Euler top, which also has a Lax representation. In addition, we obtain another Lax pair of the discrete Euler top different from both that which we derive using computer algebra and that which we introduce in terms of quaternions.*

Keywords: Nahm equation, ordinary difference equations, Lax pair, discrete matrix Euler top

1. Introduction

In the paper [4], Kamata and Nakamula described a q-analog of the Nahm equations. In this paper, we propose other discrete Nahm equations different from that in the abovementioned paper. Our discrete analog of Nahm equations includes ordinary difference equations and has a Lax representation although a Lax pair of the q-analog Nahm equations is not found. For numerical simulations, ordinary difference equations are more convenient to use than q-analog difference equations. As reduction from our discrete analog of Nahm equations, we can extend the discrete Euler top, which we introduced in [1], to a matrix version of the discrete Euler top by virtue of the key result described in [4] that the continuous Euler top can be derived from the continuous Nahm equations. Moreover, we define the equations as a discrete matrix Euler top. The discrete matrix Euler top also has a Lax representation. In addition, we obtain another Lax pair of the discrete Euler top different from both that which we derive using computer algebra [2] and that which we introduce in terms of quaternions [3].

In Section 2, we introduce a Lax pair of the continuous Nahm equations. In Section 3, we propose our discrete analog of Nahm equations and its Lax representation. In Section 4, we extend the discrete Euler top to a discrete matrix Euler top and obtain its Lax representation. In Section 5, we summarize both the Lax pair which we derive using computer algebra [2] and the Lax pair which we introduce in terms of quaternions [3]. In Section 6, we obtain another Lax pair of the discrete Euler top different from both that which we derive using computer algebra and that which we introduce in terms of quaternions. In Section 7, we discuss conserved quantities of the discrete Euler top.

2. Lax Pair of the Continuous Nahm Equations

Let $T_1(x)$, $T_2(x)$, and $T_3(x)$ be three matrix-valued meromorphic functions of a complex variable x. We define the Nahm equations by using the following equations according to the paper [5],

$$\frac{\mathrm{d}T_1}{\mathrm{d}x} = [T_2, T_3], \tag{1}$$
$$\frac{\mathrm{d}T_2}{\mathrm{d}x} = [T_3, T_1], \tag{2}$$
$$\frac{\mathrm{d}T_3}{\mathrm{d}x} = [T_1, T_2]. \tag{3}$$

Then, a Lax pair of the Nahm equations can be written as follows,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = [A, B], \tag{4}$$

where

$$A(\mu) = \begin{bmatrix} T_1(x) & -T_2(x) \\ T_2(x) & T_1(x) \end{bmatrix} + \mu \begin{bmatrix} 0 & 2T_3(x) \\ -2T_3(x) & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} T_1(x) & T_2(x) \\ -T_2(x) & T_1(x) \end{bmatrix}, \tag{5}$$

$$B(\mu) = \begin{bmatrix} 0 & T_3(x) \\ -T_3(x) & 0 \end{bmatrix} + \mu \begin{bmatrix} T_1(x) & T_2(x) \\ -T_2(x) & T_1(x) \end{bmatrix}. \tag{6}$$

As a simple consideration, the spectrum of the matrix $A(\mu)$ is well-known to not depend on x. Thus, the coefficients of the characteristic polynomial,

$$\det(\lambda I + A(\mu)), \tag{7}$$

give many conserved quantities. As a remark, the coefficients of the characteristic polynomial (7) depend on μ. However, the number of functionally independent coefficients is finite.

3. Our Discrete Analog of Nahm Equations and its Lax Representation

A q-analog of the Nahm equations is described in the paper by Kamata and Nakamula [4]. In this paper, we propose other discrete Nahm equations different from that in the paper by Kamata and Nakamula. Our discrete analog of Nahm equations includes ordinary difference equations and has a Lax representation although a Lax pair of the q-analog of the Nahm equations is not found. Let T_1^n, T_2^n, and T_3^n be three matrix-valued meromorphic functions that depend on n. We define our discrete Nahm equations by using the following equations,

$$\frac{T_1^{n+1} - T_1^n}{\delta} = T_2^{n+1}T_3^n - T_3^{n+1}T_2^n, \qquad (8)$$
$$\frac{T_2^{n+1} - T_2^n}{\delta} = T_3^{n+1}T_1^n - T_1^{n+1}T_3^n, \qquad (9)$$
$$\frac{T_3^{n+1} - T_3^n}{\delta} = T_1^{n+1}T_2^n - T_2^{n+1}T_1^n, \qquad (10)$$

where δ is a difference interval and,

$$T_1^n = T_1(n\delta), \qquad (11)$$
$$T_2^n = T_2(n\delta), \qquad (12)$$
$$T_3^n = T_3(n\delta). \qquad (13)$$

As a remark, if we take the limit of δ to 0, using eqs. (8)-(10), we can recover eqs. (1)-(3). Then, we can obtain a Lax pair of our discrete Nahm equations that can be written using the following equation,

$$A^{n+1}(\mu)B^n(\mu) = B^{n+1}(\mu)A^n(\mu), \qquad (14)$$

which is a well-known type of Lax pair described in [2], [3], [6], [7], [8]. $A^n(\mu)$ and $B^n(\mu)$ are as follows,

$$A^n(\mu) = \begin{bmatrix} T_1^n & -T_2^n \\ T_2^n & T_1^n \end{bmatrix} + \mu \begin{bmatrix} 0 & 2T_3^n \\ -2T_3^n & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} T_1^n & T_2^n \\ -T_2^n & T_1^n \end{bmatrix}, \qquad (15)$$

$$B^n(\mu) = \begin{bmatrix} -I & \delta T_3^n \\ -\delta T_3^n & -I \end{bmatrix} + \mu \begin{bmatrix} \delta T_1^n & \delta T_2^n \\ -\delta T_2^n & \delta T_1^n \end{bmatrix}. \qquad (16)$$

As a simple consideration, it is a well-known result that the eigenvalues of the characteristic polynomial,

$$p(\lambda) = \det(\lambda B^n(\mu) - A^n(\mu)), \qquad (17)$$
$$= c_m(\mu)\lambda^m + \cdots + c_0(\mu), \qquad (18)$$

where m is the matrix size of $A^n(\mu)$ and $B^n(\mu)$, do not depend on n. The result implies that we can obtain the conserved quantities using the coefficients of the characteristic polynomial (17). The conserved quantities can be defined as follows,

$$H_0(\mu) = \frac{c_0(\mu)}{c_m(\mu)}, \cdots, H_{m-1}(\mu) = \frac{c_{m-1}(\mu)}{c_m(\mu)}. \qquad (19)$$

4. Discrete Matrix Euler Top and its Lax Representation

Using the following key result described in [4] that the continuous Euler top can be derived from the continuous Nahm equations, we can extend the discrete Euler top to a discrete matrix Euler top and obtain its Lax representation. Let F_1^n, F_2^n, and F_3^n be three matrix-valued meromorphic functions that depend on n. If we set,

$$T_1^n = \begin{bmatrix} 0 & F_1^n \\ F_1^n & 0 \end{bmatrix}, \qquad (20)$$
$$T_2^n = \begin{bmatrix} 0 & -F_2^n \\ F_2^n & 0 \end{bmatrix}, \qquad (21)$$
$$T_3^n = \begin{bmatrix} F_3^n & 0 \\ 0 & -F_3^n \end{bmatrix}, \qquad (22)$$

where δ is a difference interval and,

$$F_1^n = F_1(n\delta), \qquad (23)$$
$$F_2^n = F_2(n\delta), \qquad (24)$$
$$F_3^n = F_3(n\delta), \qquad (25)$$

then, we can obtain a discrete matrix Euler top,

$$\frac{F_1^{n+1} - F_1^n}{\delta} = F_2^{n+1}F_3^n + F_3^{n+1}F_2^n, \qquad (26)$$
$$\frac{F_2^{n+1} - F_2^n}{\delta} = -(F_3^{n+1}F_1^n + F_1^{n+1}F_3^n), \qquad (27)$$
$$\frac{F_3^{n+1} - F_3^n}{\delta} = F_1^{n+1}F_2^n + F_2^{n+1}F_1^n, \qquad (28)$$

which is a matrix version of the discrete Euler top proposed in [1]. We also get a Lax pair of the discrete matrix Euler top from eqs. (15) and (16).

5. Two kinds of the Lax Pair of the Discrete Euler Top

We summarize both the Lax pair which we derive using computer algebra according to the paper[2] and the Lax pair of the discrete Euler top in terms of quaternions according to the paper[3].

5.1 Lax Pair of the Discrete Euler Top Derived Using Computer Algebra[2]

Let α_1,α_2, and α_3 be parameters. Define z_1^n, z_2^n, and z_3^n as dependent variables of n. If we define,

$$A^n = \begin{bmatrix} 0 & z_1^n & z_2^n & z_3^n \\ -z_1^n & 0 & z_3^n & -z_2^n \\ -z_2^n & -z_3^n & 0 & z_1^n \\ -z_3^n & z_2^n & -z_1^n & 0 \end{bmatrix}, \tag{29}$$

$$B^n = \begin{bmatrix} 1 & \alpha_1 z_1^n & \alpha_2 z_2^n & \alpha_3 z_3^n \\ -\alpha_1 z_1^n & 1 & \alpha_3 z_3^n & -\alpha_2 z_2^n \\ -\alpha_2 z_2^n & -\alpha_3 z_3^n & 1 & \alpha_1 z_1^n \\ -\alpha_3 z_3^n & \alpha_2 z_2^n & -\alpha_1 z_1^n & 1 \end{bmatrix}, \tag{30}$$

then we can obtain the discrete Euler top,

$$z_1^{n+1} - z_1^n = (\alpha_3 - \alpha_2)(z_2^{n+1} z_3^n + z_2^n z_3^{n+1}), \tag{31}$$
$$z_2^{n+1} - z_2^n = (\alpha_1 - \alpha_3)(z_3^{n+1} z_1^n + z_3^n z_1^{n+1}), \tag{32}$$
$$z_3^{n+1} - z_3^n = (\alpha_2 - \alpha_1)(z_1^{n+1} z_2^n + z_1^n z_2^{n+1}), \tag{33}$$

from,

$$A^{n+1} B^n = B^{n+1} A^n. \tag{34}$$

We define the discrete Euler top according to the paper[1],

$$\omega_1^{n+1} - \omega_1^n = \delta_1(\omega_2^{n+1}\omega_3^n + \omega_2^n\omega_3^{n+1}), \tag{35}$$
$$\omega_2^{n+1} - \omega_2^n = \delta_2(\omega_3^{n+1}\omega_1^n + \omega_3^n\omega_1^{n+1}), \tag{36}$$
$$\omega_3^{n+1} - \omega_3^n = \delta_3(\omega_1^{n+1}\omega_2^n + \omega_1^n\omega_2^{n+1}). \tag{37}$$

If we set,

$$z_1^n = \frac{\mathrm{i}}{\sqrt{\delta_1}}\omega_1^n, \tag{38}$$
$$z_2^n = \frac{\mathrm{i}}{\sqrt{\delta_2}}\omega_2^n, \tag{39}$$
$$z_3^n = \frac{\sqrt{2}}{\sqrt{\delta_3}}\omega_3^n, \tag{40}$$
$$\alpha_1 = \frac{\sqrt{\delta_1}\sqrt{\delta_2}\sqrt{\delta_3}}{\sqrt{2}}, \tag{41}$$
$$\alpha_2 = \frac{-\sqrt{\delta_1}\sqrt{\delta_2}\sqrt{\delta_3}}{\sqrt{2}}, \tag{42}$$
$$\alpha_3 = 0, \tag{43}$$

we can transform eqs. (31)-(33) into eqs. (35)-(37).

5.2 Lax Pair of the Discrete Euler Top in Terms of Quaternions[3]

Let β_1,β_2, and β_3 be parameters. Define y_1^n, y_2^n, and y_3^n as dependent variables of n. If we define,

$$A^n = \begin{bmatrix} 0 & y_1^n & y_2^n & y_3^n \\ -y_1^n & 0 & -y_3^n & y_2^n \\ -y_2^n & y_3^n & 0 & -y_1^n \\ -y_3^n & -y_2^n & y_1^n & 0 \end{bmatrix}, \tag{44}$$

$$B^n = \begin{bmatrix} 1 & \beta_1 y_1^n & \beta_2 y_2^n & \beta_3 y_3^n \\ -\beta_1 y_1^n & 1 & -\beta_3 y_3^n & \beta_2 y_2^n \\ -\beta_2 y_2^n & \beta_3 y_3^n & 1 & -\beta_1 y_1^n \\ -\beta_3 y_3^n & -\beta_2 y_2^n & \beta_1 y_1^n & 1 \end{bmatrix}, \tag{45}$$

then we can obtain the discrete Euler top,

$$y_1^{n+1} - y_1^n = (\beta_2 - \beta_3)(y_2^{n+1} y_3^n + y_2^n y_3^{n+1}), \tag{46}$$
$$y_2^{n+1} - y_2^n = (\beta_3 - \beta_1)(y_3^{n+1} y_1^n + y_3^n y_1^{n+1}), \tag{47}$$
$$y_3^{n+1} - y_3^n = (\beta_1 - \beta_2)(y_1^{n+1} y_2^n + y_1^n y_2^{n+1}), \tag{48}$$

from eq. (34). If we restrict β_1,β_2,β_3,y_1^0, y_2^0, and y_3^0 to real numbers, then we can rewrite A^n and B^n as follows,

$$A^n = ((y_1^n)\mathbf{i} + (y_2^n)\mathbf{j} + (y_3^n)\mathbf{k}), \tag{49}$$
$$B^n = (1 + (\beta_1 y_1^n)\mathbf{i} + (\beta_2 y_2^n)\mathbf{j} + (\beta_3 y_3^n)\mathbf{k}), \tag{50}$$

where **i**, **j**, and **k** are the fundamental quaternion units.

If we set,

$$y_1^n = \frac{\mathrm{i}}{\sqrt{\delta_1}}\omega_1^n, \tag{51}$$
$$y_2^n = \frac{\mathrm{i}}{\sqrt{\delta_2}}\omega_2^n, \tag{52}$$
$$y_3^n = \frac{\sqrt{2}}{\sqrt{\delta_3}}\omega_3^n, \tag{53}$$
$$\beta_1 = \frac{-\sqrt{\delta_1}\sqrt{\delta_2}\sqrt{\delta_3}}{\sqrt{2}}, \tag{54}$$
$$\beta_2 = \frac{\sqrt{\delta_1}\sqrt{\delta_2}\sqrt{\delta_3}}{\sqrt{2}}, \tag{55}$$
$$\beta_3 = 0, \tag{56}$$

we can transform eqs. (46)-(48) into eqs. (35)-(37).

6. Another Lax Pair of the Discrete Euler Top

In addition, we propose another Lax pair of the discrete Euler top different from that in the paper [3]. Define x_1^n, x_2^n, and x_3^n as scalar dependent variables of n. If we set,

$$F_1^n = x_1^n, \tag{57}$$
$$F_2^n = x_2^n, \tag{58}$$
$$F_3^n = x_3^n, \tag{59}$$

where δ is a difference interval and,

$$x_1^n = x_1(n\delta), \quad (60)$$
$$x_2^n = x_2(n\delta), \quad (61)$$
$$x_3^n = x_3(n\delta), \quad (62)$$

and define,

$$A^n(\mu) = \begin{bmatrix} 0 & x_1^n & 0 & x_2^n \\ x_1^n & 0 & -x_2^n & 0 \\ 0 & -x_2^n & 0 & x_1^n \\ x_2^n & 0 & x_1^n & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 2x_3^n & 0 \\ 0 & 0 & 0 & -2x_3^n \\ -2x_3^n & 0 & 0 & 0 \\ 0 & 2x_3^n & 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} 0 & x_1^n & 0 & -x_2^n \\ x_1^n & 0 & x_2^n & 0 \\ 0 & x_2^n & 0 & x_1^n \\ -x_2^n & 0 & x_1^n & 0 \end{bmatrix}, \quad (63)$$

$$B^n(\mu) = \begin{bmatrix} -1 & 0 & \delta x_3^n & 0 \\ 0 & -1 & 0 & -\delta x_3^n \\ -\delta x_3^n & 0 & -1 & 0 \\ 0 & \delta x_3^n & 0 & -1 \end{bmatrix} + \mu \begin{bmatrix} 0 & \delta x_1^n & 0 & -\delta x_2^n \\ \delta x_1^n & 0 & \delta x_2^n & 0 \\ 0 & \delta x_2^n & 0 & \delta x_1^n \\ -\delta x_2^n & 0 & \delta x_1^n & 0 \end{bmatrix}, \quad (64)$$

then, we can obtain the discrete Euler top,

$$\frac{x_1^{n+1} - x_1^n}{\delta} = x_2^{n+1} x_3^n + x_3^{n+1} x_2^n, \quad (65)$$
$$\frac{x_2^{n+1} - x_2^n}{\delta} = -(x_3^{n+1} x_1^n + x_1^{n+1} x_3^n), \quad (66)$$
$$\frac{x_3^{n+1} - x_3^n}{\delta} = x_1^{n+1} x_2^n + x_2^{n+1} x_1^n, \quad (67)$$

from eq. (14).

7. Conserved Quantities of the Discrete Euler Top

Hereafter, we set $\delta = 1$. From $H_0(0)$ and $H_3(1)$ in eq. (19), we can get two conserved quantities of the discrete Euler top. We can check that $H_0(0)$ and $H_3(1)$ are equivalent to the conserved quantities described in [1] as follows. If we set,

$$x_1^n = \mathrm{i}\sqrt{\delta_2}\sqrt{\delta_3}\omega_1^n, \quad (68)$$
$$x_2^n = \sqrt{\delta_1}\sqrt{\delta_3}\omega_2^n, \quad (69)$$
$$x_3^n = \mathrm{i}\sqrt{\delta_1}\sqrt{\delta_2}\omega_3^n, \quad (70)$$

we can transform eqs. (65)-(67) into eqs. (35)-(37). Moreover, we can obtain two conserved quantities $H_0'(0)$ and $H_3'(1)$ of eqs. (35)-(37),

$$H_0'(0) = \delta_3^2 \left(\frac{\delta_1(\omega_2^n)^2 - \delta_2(\omega_1^n)^2}{\delta_1\delta_2(\omega_3^n)^2 - 1} \right)^2, \quad (71)$$
$$H_3'(1) = \frac{8\delta_2(\delta_1(\omega_3^n)^2 - \delta_3(\omega_1^n)^2)}{\delta_2\delta_3(\omega_1^n)^2 - \delta_1\delta_3(\omega_2^n)^2 - \delta_1\delta_2(\omega_3^n)^2 + 1}, \quad (72)$$

from $H_0(0)$ and $H_3(1)$ of the eqs. (65)-(67),

$$H_0(0) = \left(\frac{(x_1^n)^2 + (x_2^n)^2}{(x_3^n)^2 + 1} \right)^2, \quad (73)$$
$$H_3(1) = \frac{8(x_1^n + x_3^n)(x_1^n - x_3^n)}{-(x_1^n)^2 - (x_2^n)^2 + (x_3^n)^2 + 1}, \quad (74)$$

using the eqs. (68)-(70). Thus, G_1 and G_2 are conserved quantities,

$$G_1 = \frac{\delta_1(\omega_2^n)^2 - \delta_2(\omega_1^n)^2}{1 - \delta_1\delta_2(\omega_3^n)^2}, \quad (75)$$
$$G_2 = \frac{\delta_1(\omega_3^n)^2 - \delta_3(\omega_1^n)^2}{\delta_2\delta_3(\omega_1^n)^2 - \delta_1\delta_3(\omega_2^n)^2 - \delta_1\delta_2(\omega_3^n)^2 + 1}, \quad (76)$$

because,

$$H_0'(0) = \delta_3^2 G_1^2, \quad (77)$$
$$H_3'(1) = 8\delta_2 G_2. \quad (78)$$

In [1], we get three conserved quantities,

$$G_3 = \frac{\delta_1(\omega_2^n)^2 - \delta_2(\omega_1^n)^2}{1 - \delta_1\delta_2(\omega_3^n)^2}, \quad (79)$$
$$G_4 = \frac{\delta_2(\omega_3^n)^2 - \delta_3(\omega_2^n)^2}{1 - \delta_2\delta_3(\omega_1^n)^2}, \quad (80)$$
$$G_5 = \frac{\delta_3(\omega_1^n)^2 - \delta_1(\omega_3^n)^2}{1 - \delta_3\delta_1(\omega_2^n)^2}, \quad (81)$$

where $G_1 = G_3$. If we define matrices J_1, J_2 and J_3 as follows,

$$J_1 = \begin{bmatrix} \frac{\partial G_1}{\partial \omega_1^n} & \frac{\partial G_1}{\partial \omega_2^n} & \frac{\partial G_1}{\partial \omega_3^n} \\ \frac{\partial G_2}{\partial \omega_1^n} & \frac{\partial G_2}{\partial \omega_2^n} & \frac{\partial G_2}{\partial \omega_3^n} \end{bmatrix}, \quad (82)$$
$$J_2 = \begin{bmatrix} \frac{\partial G_3}{\partial \omega_1^n} & \frac{\partial G_3}{\partial \omega_2^n} & \frac{\partial G_3}{\partial \omega_3^n} \\ \frac{\partial G_4}{\partial \omega_1^n} & \frac{\partial G_4}{\partial \omega_2^n} & \frac{\partial G_4}{\partial \omega_3^n} \\ \frac{\partial G_5}{\partial \omega_1^n} & \frac{\partial G_5}{\partial \omega_2^n} & \frac{\partial G_5}{\partial \omega_3^n} \end{bmatrix}, \quad (83)$$
$$J_3 = \begin{bmatrix} \frac{\partial G_3}{\partial \omega_1^n} & \frac{\partial G_3}{\partial \omega_2^n} & \frac{\partial G_3}{\partial \omega_3^n} \\ \frac{\partial G_4}{\partial \omega_1^n} & \frac{\partial G_4}{\partial \omega_2^n} & \frac{\partial G_4}{\partial \omega_3^n} \end{bmatrix}, \quad (84)$$

then, the ranks of J_1, J_2 and J_3 are 2,2 and 2, respectively. In addition, we can get,

$$\begin{vmatrix} \frac{\partial G_1}{\partial \omega_1^n} & \frac{\partial G_1}{\partial \omega_2^n} & \frac{\partial G_1}{\partial \omega_3^n} \\ \frac{\partial G_2}{\partial \omega_1^n} & \frac{\partial G_2}{\partial \omega_2^n} & \frac{\partial G_2}{\partial \omega_3^n} \\ \frac{\partial G_4}{\partial \omega_1^n} & \frac{\partial G_4}{\partial \omega_2^n} & \frac{\partial G_4}{\partial \omega_3^n} \end{vmatrix} = 0, \qquad (85)$$

$$\begin{vmatrix} \frac{\partial G_3}{\partial \omega_1^n} & \frac{\partial G_3}{\partial \omega_2^n} & \frac{\partial G_3}{\partial \omega_3^n} \\ \frac{\partial G_4}{\partial \omega_1^n} & \frac{\partial G_4}{\partial \omega_2^n} & \frac{\partial G_4}{\partial \omega_3^n} \\ \frac{\partial G_2}{\partial \omega_1^n} & \frac{\partial G_2}{\partial \omega_2^n} & \frac{\partial G_2}{\partial \omega_3^n} \end{vmatrix} = 0. \qquad (86)$$

Therefore, $H_0(0)$ and $H_3(1)$ are equivalent to the conserved quantities G_3,G_4, and G_5 described in [1].

8. Conclusion

In this paper, we proposed our discrete Nahm equations different from that in the paper by Kamata and Nakamula. Our discrete analog of Nahm equations includes ordinary difference equations and has a Lax representation. As reduction from our discrete analog of Nahm equations, we extended the discrete Euler top, which we introduced in [1], to a discrete matrix Euler top. In addition, we obtain another Lax pair of the discrete Euler top different from both that which we derive using computer algebra [2] and that which we introduce in terms of quaternions [3].

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