A Parallel Method for the Eigenpairs of the Generalized Eigenvalue Problem

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***Abstract*— A parallel method for approximating the eigenpairs of the generalized eigenvalue problem $Ax = \lambda Bx$ is presented. The method is simple and based on solving quadratic systems. Newton's method is used to find the solution of the nonlinear quadratic system. To ensure an accurate starting vector for Newton's method, a homotopy method is used together with Gerschgorin circle theorem. Since the method works in parallel, its efficiency can be increased many times if implemented on computers with parallel architecture.The method has advantage of working in parallel. The method described and numerical examples are given.**

Keywords: Generalized, Eigenvalues, Quadratic, Parallel.

I. Introduction

The purpose of this paper is to study the numerical approximation of the general eigenvalue problem

$$A\mathbf{x} = \lambda B\mathbf{x} \tag{1}$$

where here A is assumed to be symmetric, and B symmetric and positive definite. This problem arises in many areas in engineering and science. For example, in the modeling of aerodynamical systems, in the equations of motions of a mechanical system under the application of a conservative system of forces, vibrating mechanical systems (see [1]) and many others. A popular method for solving problem (1) is by reduction to a standard symmetric eigenvalue problem. That is by replacing B with LL^T where L is a real, nonsingular lower triangular matrix (see [2]). A serious drawback of this method is that if A and B are sparse, they will lose their sparsity in the reduction process. Hence the algorithm requires the explicit storage of the entire matrix. In ([3]) the quadratics method was used to solve the nongeneral eigenvalue problem $Ax = \lambda x$.

In this paper we propose a new method for solving problem (1) that has the advantage of working in parallel. The method is simple and based on solving quadratic systems (see [3]). Newton's method is used to find the solution of the nonlinear quadratic system. To ensure an accurate starting vector for Newton's method, a homotopy method is used together with Gerschgorin circle theorem. Since the method works in parallel, its efficiency can be increased many times if implemented on computers with parallel architecture.

II. The Numerical Method

Following ([3]) let A be $n \times n$ symmetric matrix and B be the an $n \times n$ symmetric and positive definite matrix. Let X_i be the eigenvector of A which has 1 in the ith position and λ_i its corresponding eigenvalue. Then the algebraic eigenvalue problem

$$AX_i = \lambda_i BX_i \tag{2}$$

is a nonlinear system of n equations in n unknowns: λ_i, x_i, $i = 1, 2, ..., i-1, i+1, ..., n$.

The ith eqn. of (2) is given by

$$a_{ii} + \sum_{p=1}^{i-1} a_{ip}x_p + \sum_{p=i+1}^{n} a_{ip}x_p = \lambda_i(b_{ii} + \sum_{p=1}^{i-1} b_{ip}x_p + \sum_{p=i+1}^{n} b_{ip}x_p) \tag{3}$$

When $j \neq i$ the jth eqn. of (2) is

$$a_{ji} + \sum_{p=1}^{i-1} a_{jp}\,x_p + \sum_{p=i+1}^{n} a_{jp}\,x_p = \lambda_i(b_{ji} + \sum_{p=1}^{i-1} b_{jp}x_p + \sum_{p=i+1}^{n} b_{jp}x_p) \tag{4}$$

Using (3) and (4) we obtain, for $j = 1, ..., i-1, i+1, ..., n$

$$\begin{aligned}(a_{ji} + \sum_{p=1}^{i-1} a_{jp}x_p + \sum_{p=i+1}^{n} a_{jp}x_p)(b_{ii} + \sum_{p=1}^{i-1} b_{ip}x_p + \sum_{p=i+1}^{n} b_{ip}x_p) \\ = (a_{ii} + \sum_{p=1}^{i-1} a_{ip}x_p + \sum_{p=i+1}^{n} a_{ip}x_p\, x_i)\cdot \\ (b_{ji} + \sum_{p=1}^{i-1} b_{jp}x_p + \sum_{p=i+1}^{n} b_{jp}x_p)\end{aligned} \tag{5}$$

For $j = 1, ..., i-1$ (5) takes the form

$$\begin{aligned}(b_{ii} + b_{ij}x_j + \sum_{p=1}^{j-1} b_{ip}x_p + \sum_{p=j+1}^{i-1} b_{ip}x_p + \sum_{p=i+1}^{n} b_{ip}x_p)\cdot \\ (a_{ji} + a_{jj}x_j + \sum_{p=1}^{j-1} a_{jp}x_p + \sum_{p=j+1}^{i-1} a_{jp}x_p + \sum_{p=i+1}^{n} a_{jp}x_p) = \\ (a_{ii} + a_{ij}x_j + \sum_{p=1}^{j-1} a_{ip}x_p + \sum_{p=j+1}^{i-1} a_{ip}x_p + \sum_{p=i+1}^{n} a_{ip}x_p)\cdot \\ (b_{ji} + b_{jj}x_j + \sum_{p=1}^{j-1} b_{jp}x_p + \sum_{p=j+1}^{i-1} b_{jp}x_p + \sum_{p=i+1}^{n} b_{jp}x_p)\end{aligned} \tag{6}$$

and for $j = i+1, ..., n$

$$\begin{aligned}&(b_{ii}+b_{ij}x_j+\sum_{p=1}^{i-1} b_{ip}x_p+\sum_{p=j+1}^{j-1} b_{ip}x_p+\sum_{p=i+1}^{n} b_{ip}x_p)\cdot\\ &(a_{ji}+a_{jj}x_j+\sum_{p=1}^{i-1} a_{jp}x_p+\sum_{p=j+1}^{j-1} a_{jp}x_p+\sum_{p=i+1}^{n} a_{jp}x_p)=\\ &(a_{ii}+a_{ij}x_j+\sum_{p=1}^{i-1} a_{ip}x_p+\sum_{p=j+1}^{j-1} a_{ip}x_p+\sum_{p=i+1}^{n} a_{ip}x_p)\cdot\\ &(b_{ji}+b_{jj}x_j+\sum_{p=1}^{i-1} b_{jp}x_p+\sum_{p=j+1}^{j-1} b_{jp}x_p+\sum_{p=i+1}^{n} b_{jp}x_p)\end{aligned} \quad (7)$$

Let us define the notation

$$\begin{aligned}(H_j^i)_i &= \sum_{p=1}^{j-1} H_{ip}x_p+\sum_{p=j+1}^{i-1} H_{ip}x_p+\sum_{p=i+1}^{n} H_{ip}x_p \quad (8)\\ (H_i^j)_i &= \sum_{p=1}^{i-1} H_{ip}x_p+\sum_{p=j+1}^{j-1} H_{ip}x_p+\sum_{p=i+1}^{n} H_{ip}x_p\end{aligned}$$

By using the above notation, Eqns (6) and (7) take the form of the quadratic system

$$f_j = A_j x_j^2 + B_j x_j + C_j = 0 \quad (9)$$

where for $j = 1, ..., i-1$

$$\begin{aligned}A_j &= b_{ij}a_{jj}-a_{ij}b_{jj}\\ B_j &= b_{ii}a_{jj}-a_{ii}b_{jj}+b_{ij}a_{ji}-b_{ji}a_{ij}+(b_j^i)_i a_{jj}-\\ &\quad (a_j^i)_i b_{jj}+(a_j^i)_j b_{ij}-(b_j^i)_j a_{ij}\\ C_j &= b_{ii}a_{ji}+b_{ii}(a_j^i)_j+a_{ji}(b_j^i)_i+(b_j^i)_i(a_j^i)_j-\\ &\quad a_{ii}b_{ji}-a_{ii}(b_j^i)_j-b_{ji}(a_j^i)_i-(a_j^i)_i(b_j^i)_j\end{aligned}$$

and for $j = i+1, ..., n$

$$\begin{aligned}A_j &= b_{ij}a_{jj} - a_{ij}b_{jj}\\ B_j &= b_{ii}a_{jj}-a_{ii}b_{jj}+b_{ij}a_{ji}-b_{ji}a_{ij}+(b_i^j)_i a_{jj}-\\ &\quad (a_i^j)_i b_{jj}+(a_i^j)_j b_{ij}-(b_i^j)_j a_{ij}\\ C_j &= b_{ii}a_{ji}+b_{ii}(a_i^j)_j+a_{ji}(b_i^j)_i+(b_i^j)_i(a_i^j)_j-\\ &\quad a_{ii}b_{ji}-a_{ii}(b_i^j)_j-b_{ji}(a_i^j)_i-(a_i^j)_i(b_i^j)_j\end{aligned}$$

The nonlinear system (9) can be solved by using Newton's method. The $(n-1)\times(n-1)$ Jacobian matrix $J = (q_{lm})$ of the system is defined by

$$(q_{lm}) = \begin{cases} 2A_l x_l + B_l & \text{if } l = m\\ E_{lm} & \text{if } l \neq m\end{cases}. \quad (10)$$

where

$$\begin{aligned}E_{lm} &= (a_{lm}b_{il}-a_{il}b_{lm}+a_{ll}b_{im}-a_{im}b_{ll})x_l+a_{lm}b_{ii}-\\ &\quad a_{ii}b_{lm}+a_{li}b_{im}-a_{im}b_{li}+a_{lm}b_{im}-a_{im}b_{lm}\end{aligned}$$

The functional iteration procedure evolves from selecting $\mathbf{x}^{(0)}$ and generating for $k \geq 1$,

$$\mathbf{x}^{(k)} = \mathbf{x}^{(k-1)} + J(\mathbf{x}^{(k-1)})^{-1}\mathbf{F}(\mathbf{x}^{(k-1)}) \quad (11)$$

where $F = -(f_1, ..., f_{i-1}, f_{i+1}, ..., f_n)$.

Once the eigenvectors are obtained from (11), the corresponding eigenvalues are readily obtained from Eq. (3).

It is well know that Newton's method is generally expected to give quadratic convergence, provided that a sufficiently accurate starting value is known and $J(\mathbf{x})^{-1}$ exists. To insure an accurate starting value, we introduce a homotopy method which consists of a sequence of matrices defined by

$$\begin{aligned}A_k &= D_1+t_kP, \quad 0 \leq t_k \leq 1 \quad (12)\\ B_k &= D_2+t_kQ, \quad 0 \leq t_k \leq 1\end{aligned}$$

where $t_k = kh,\ k = 0, 1, .., N,\ N = 1/h$, and

$$D_1 = \begin{bmatrix} a_{11} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & a_{nn}\end{bmatrix},$$

$$t_kP = \begin{bmatrix} 0 & t_ka_{12} & \cdots & t_ka_{1n}\\ t_ka_{21} & 0 & \cdots & t_ka_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ t_ka_{n1} & \cdots & t_ka_{n,n-1} & 0\end{bmatrix}$$

$$D_2 = \begin{bmatrix} b_{11} & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & b_{nn}\end{bmatrix},$$

$$t_kQ = \begin{bmatrix} 0 & t_kb_{12} & \cdots & t_kb_{1n}\\ t_kb_{21} & 0 & \cdots & t_kb_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ t_kb_{n1} & \cdots & t_kb_{n,n-1} & 0\end{bmatrix}$$

Note that $A_0 = D_1$, $B_0 = D_2$ and $A_N = A$, $B_N = B$. One way of selecting h is by using the Gerschgorin circle theorem. Choose $h_G = 2^{-s}$ so that all the circles of A_1 are disjoint with s given by

$$\left[\left[\frac{\ln(r/d)}{\ln 2}+2\right]\right], \quad \text{where} \quad (13)$$

$$r = \max_i\{\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|/b_{ii}\} \text{ and} \quad (14)$$

$$d = \min_{\substack{i\leq j\leq n\\ j\neq i}}\{|a_{ii}/b_{ii} - a_{jj}/b_{jj}|\} \quad (15)$$

Thus, given h_G, the method consists in finding the sequence of eigenpairs of $A_0\mathbf{x}^{(0)} = \lambda B_0\mathbf{x}^{(0)}, ..., A_N\mathbf{x}^{(N)} = \lambda B_N\mathbf{x}^{(N)}$ obtained by solving for each matrix A_i and B_i the nonlinear system (9) using Newton's method.

The steps for finding the starting value for the application of Newton's method are:

At t_0 the eigenvalues are known and given by the diagonal elements of $B_0^{-1}A_0$. That is $\lambda_i^{(0)} = \{a_{ii}/b_{ii}\}$ and the corresponding eigenvectors are

$$x^{(0)} = \left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, ..., \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right).$$

At t_1 Newton's method is applied to $A_1\mathbf{x}^{(1)} = \lambda B_1\mathbf{x}^{(1)}$ using as a starting value the above eigenvectors to get $\mathbf{x}^{(1)}$. At t_2, the two point interpolation formula

$$\mathbf{y}^{(2)} = 2\mathbf{x}^{(1)} - \mathbf{x}^{(0)} \tag{16}$$

is used to find the starting value $\mathbf{y}^{(2)}$. Here $\mathbf{x}^{(0)}$ and $\mathbf{x}^{(1)}$ are the eigenvectors of (A_0, B_0) and (A_1, B_1) respectively.

For $t_k > t_2$, the three previous eigenvectors $\mathbf{x}^{(k-1)}$, $\mathbf{x}^{(k-2)}$, $\mathbf{x}^{(k-3)}$ of (A_{k-1}, B_{k-1}), (A_{k-2}, B_{k-2}), (A_{k-3}, B_{k-3}) respectively are used together with the three point interpolation formula

$$\mathbf{y}^{(k)} = 3\mathbf{x}^{(k-1)} - 3\mathbf{x}^{(k-2)} + \mathbf{x}^{(k-3)} \tag{17}$$

to find the starting value $\mathbf{y}^{(k)}$ of Newton's method to compute the eigenpairs of (A_k, B_k). So the method consists on finding the eigenpairs of the sequence of the generalized eigenvalue problems $A_0x^{(0)} = \lambda B_0x^{(0)}, ..., A_Nx^{(N)} = \lambda B_Nx^{(N)}$ by solving at each step a quadratic nonlinear system along with formula (17) to get the starting values.

III. Numerical Results

The following numerical results were performed. As mentioned above, if the numerical examples were performed in PC with many processors, each eigenpair can be assigned to a separate processor.

Example 1: *Consider the generalized eigenvalue problem given by*

$$A = \begin{bmatrix} -153 & -39 & 3 & 49 & -28 \\ -39 & -74 & -12 & 55 & 1 \\ 3 & -12 & 60 & -20 & 43 \\ 49 & 55 & -20 & 157 & -27 \\ -28 & 1 & 43 & -27 & 1 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 180 & 32 & 32 & 9 & 25 \\ 32 & 68 & 64 & 6 & 22 \\ 32 & 64 & 135 & 42 & 25 \\ 9 & 6 & 42 & 32 & 42 \\ 25 & 22 & 25 & 42 & 197 \end{bmatrix}$$

Note that the algorithm described above gives only the eigenpairs of $A_k\mathbf{x} = \lambda B_k\mathbf{x}$ that are real. The fact that A and B are symmetric does not mean that the eigenpairs are real since the matrix $B^{-1}A$ is in general nonsymmetric.

The application of the algorithm gives the eigenvalues

$$\begin{aligned} \lambda_1 &= -0.822825618207529 \\ \lambda_2 &= -1.780142517580858 \\ \lambda_3 &= 0.437528154079936 \\ \lambda_4 &= 40.446810470397770 \\ \lambda_5 &= -0.093378396611026 \end{aligned}$$

which are in perfect agreements with the eigenpairs obtained from MATLAB.

Example 2: *Consider the vibrating mechanical system*

$$Mq'' + Cq' + Kq = 0$$

where M, C,and K are $n \times n$ matrices and $q = q(t)$ is a vector of length n.

If one seeks a solution in the form $q = ae^{rt}$, then one is led to an eigenvalue type problem of the form

$$(Mr^2 + Cr + K)a = 0$$

that is, one seeks values of r for which there are nonzero vectors for solving the linear system. M is the so-called mass matrix and is always positive definite and depending on the constraints, C is nonnegative definite and K is positive definite. In the event where there is no friction, then $C = 0$ and the system reduces to the generalized eigenvalue problem (1) with $B = -M$, $A = K$ and $\lambda = r^2$.

As an example let us consider the double pendulum. Consider the motion of the two masses m_1 and m_2 (see Fig. 1).

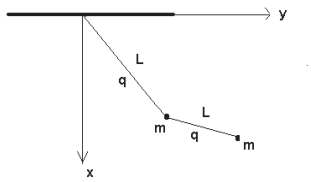

Fig. 1. The double pendulum

If no friction is present, then $C = 0$ and M and K are defined by

$$M = \begin{bmatrix} m_1L_1^2 + m_2L_1^2 & m_1L_1L_2 \\ m_1L_1L_2 & m_2L_2 \end{bmatrix} \text{ and}$$

$$K = \begin{bmatrix} (m_1 + m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{bmatrix}$$

where g is the acceleration due to gravity, L_1 and L_2 as shown in Fig. 1.

If we set $L_1 = 3$ m, $L_2 = 1$ m, $m_1 = 2$ kg, $m_2 = 3$ kg and $g = 9.8$ m/sec^2, then

$$B = -M = \begin{bmatrix} -21 & -6 \\ -6 & -3 \end{bmatrix} \text{ and } A = K = \begin{bmatrix} 147.0 & 0 \\ 0 & 29.4 \end{bmatrix}.$$

Using the algorithm described above we get

$$\lambda_1 = -4.63025 \text{ and } \lambda_2 = -34.56975.$$

IV. Conclusion

We consider in this paper the numerical approximation of the generalized eigenvalue problem. We present a new method for computing the eigenpairs of the generalized eigenvalue problem. The eigenpairs of the problem are obtained from the solution of a simple quadratic system. We used Newton's method along with a homotopy method to solve the nonlinear quadratic system. Finally, numerical examples were provided to illustrate the working procedure of the new method and the accuracy of the numerical procedure was demonstrated by comparing with results obtained from MATLAB.

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