On the Construction of Optimal Node-Disjoint Paths in Folded Hypercubes of Odd Dimensions

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Abstract – *Node-disjoint paths have made significant contributions to the study of routing, reliability, and fault tolerance of an interconnection network. In this paper, we construct m node-disjoint paths from one source node to other m (not necessarily distinct) target nodes, respectively, in an n-dimensional folded hypercube so that not only is their total length minimized, but their maximal length is also minimized in the worst case, where* $m \le n+1$ *and n is odd. In addition, these m node-disjoint paths can be constructed in* $O(m^2 n^{2.5})$ *time and each path is either shortest or nearly shortest.*

Keywords: Folded hypercube, hypercube, node-disjoint paths, matching, optimization

1 Introduction

Advanced hardware technology has made it easy to build a large-scale multiprocessor system consisting of hundreds or even thousands of processors. Before designing a multi-processor system, its interconnection network (network for short) in which nodes and links, respectively, correspond to processors and communication channels must be determined first. Since the topology of a network makes great influences in the system performance, many possible options have been proposed in the literature for practical implementations and/or theoretical studies. For the latter purpose, the folded hypercube was one of the networks which have received much attention from outstanding researchers [5-7, 12-15, 19, 22, 23, 26, 29-31].

A folded hypercube is basically a hypercube with additional links augmented, where the additional links connect all pairs of nodes whose distances are longest in the hypercube. An n-dimensional hypercube (abbreviated to an n-cube) [25] consists of 2^n nodes that are labeled with 2^n binary strings of length n. Two nodes are connected by a link if and only if they differ by exactly one bit. The diameter of an n-cube is n. On the other hand, an n-dimensional folded hypercube (abbreviated to an n-fcube) [5] is basically an n-cube augmented with 2^{n-1} complement links. Each *complement link* connects two nodes whose labels are the complements of each other. Figure 1 shows the structure of a 3-fcube, where (000, 111), (001, 110), (010, 101), and (011, 100) are the four complement links. With the help of complement links, the diameter of an n-fcube is reduced to $\lceil n/2 \rceil$. An n-cube and an n-fcube have connectivities n and n+1, respectively. The *connectivity* of a network is the minimum number of nodes whose removal can make the network disconnected or trivial.

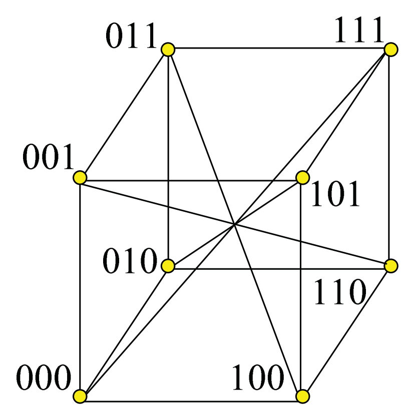

Figure 1. The structure of a 3-fcube.

A (simple) path in a network consists of a sequence of mutually distinct nodes such that there is a link between any two consecutive nodes. Two paths are *internally node-disjoint* (disjoint for short) if they do not share any common node except their end nodes. Disjoint paths have made themselves play an significant role in the study of routing, reliability, and fault tolerance of a network because we can use them to avoid congestion, accelerate transmission rate, and provide alternative transmission routes. There are three kinds of disjoint paths [3], i.e., one-to-one [9, 27], one-to-many [1, 4, 9, 11, 18, 21, 28], and many-to-many [2, 10, 11]. The one-to- one disjoint paths, also called the *container* [9], have common end nodes. According to Menger's theorem, there exists a container for any two nodes in a network with its width, i.e. the number of disjoint paths in it, not less than the connectivity of the network. The study of containers provides important measures, such as wide-distance and

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wide-diameter, for analyzing the reliability and fault tolerance of a network.

The one-to-many disjoint paths from a common node to other mutually distinct nodes were first studied in [24], and an Information Dispersal Algorithm (IDA for short) was proposed on the hypercube. By taking advantages of disjoint paths, the IDA has numerous potential applications to secure and fault-tolerant storage and transmission of information. In addition, the study of one-to-many disjoint paths provides important measures, such as star-diameter and Rabin number, for analyzing the reliability and fault tolerance of a network. As described literally, the many-to-many disjoint paths connect two sets of nodes. In order to reduce the transmission cost and latency, the total length and maximal length of disjoint paths are required to be minimized, respectively, where the *length* of a path is the number of links in it.

Routing functions have been shown to be effective on deriving various disjoint paths in the hypercube [16] and its variants, such as tori [17], generalized hypercubes [18], and folded hypercubes [19]. In this paper, we study the problem of constructing m disjoint paths from one source node to other m (not necessarily distinct) target nodes, respectively, in an n-fcube so that not only is their total length minimized, but their maximal length is also minimized in the worst case, where $m \leq n+1$ and n is odd. By adopting the idea similar to [19], this problem was first transformed into a corresponding problem of constructing disjoint paths in an $(n+1)$-cube with special properties, and then its solutions are applied to derive the required paths (in the n-fcube). By the aid of the routing functions and construction procedures in [16], it will be shown that these optimal m disjoint paths can be constructed in $O(m^2n^{2.5})$ time. Since IDA [24] relies heavily on one-to-many disjoint paths, the construction of optimal disjoint paths in an n-fcube is not only theoretically interesting but also practical in real applications.

The rest of this paper is organized as follows: In the next section, the routing functions and the construction of disjoint shortest paths in an n-cube are described. Both of them are necessarily used in this paper. As shown in Section 3, the optimal disjoint paths in an n-fcube can be constructed in $O(m^2n^{2.5})$ time. In Section 4, this paper concludes with some remarks on the efficiency of our results and describes the future work. For the brevity of this paper, whenever we discuss time complexity, we mean worst-case time complexity.

2 Preliminary

In this section, we briefly describe some results of [16] which are necessarily used in this paper. For referential integrity, we also follow the symbols in [16]. Suppose that s is the source node and $d_1, d_2, \ldots, d_m$ are m (not necessarily distinct) destination nodes in an n-cube, where $m \leq n$ and $s \notin \{d_1, d_2, \ldots, d_m\}$. Since an n-cube is node symmetric, we assume $s = \overbrace{00\ldots0}^{n} = 0^n$, i.e. the origin, without loss of generality. Let $D=\{d_1, d_2, \ldots, d_m\}$ be a multiset and $I=\{k_1, k_2, \ldots, k_m\}$ be a set of m distinct integers ranging from 1 to n (actually, $k_1, k_2, \ldots, k_m$ denote m dimensions of an n-cube). A *multiset* is a collection of elements in which multiple occurrences of the same element are allowed. In [16], a one-to-one correspondence $\Phi: D \to I$ was referred to as a *routing function*, and it was shown that routing functions can be effectively used to derive m disjoint paths from s to $d_1, d_2, \ldots, d_m$, respectively, in an n-cube.

As defined in [16], we let $e_{k_t} = 0^{k_t-1}10^{n-k_t}$ and $d_i = d_{i,1}d_{i,2}\bullet\bullet\bullet d_{i,n}$, where $1 \leq t \leq m$, $1 \leq i \leq m$, and $d_{i,k}$ denotes the kth bit (from the left) of d_i for all $1 \leq k \leq n$. Intuitively, $\Phi(d_i)=k_t$ means that e_{k_t} is chosen for the node following s when we route from s to d_i. Since $d_{i,\Phi(d_i)}=d_{i,k_t}=1$ assures that e_{k_t} is included in a shortest path from s to d_i, we prefer a Φ with $d_{i,\Phi(d_i)}=1$ for all $1 \leq i \leq m$ in order to route a shortest path from s to d_i. Unfortunately, the preferred Φ does not always exist for arbitrary D and I.

In [16], an optimal $O(mn)$ construction procedure named *Paths1* was proposed. With input arguments Φ, m, n, D, and I, it was shown in [16] that *Paths1*(Φ, m, n, D, I) can produce m disjoint paths, denoted by $Q_1, Q_2, \ldots, Q_m$, from s to $d_1, d_2, \ldots, d_m$, respectively, so that Q_i is shortest with length $|d_i|$ if $d_{i,\Phi(d_i)}=1$ for all $1 \leq i \leq m$, where $|d_i|$ denotes the number of bits 1 contained in d_i, i.e. the distance from s to d_i. The following lemma describes the above results formally.

Lemma 1.[16] Given a routing function $\Phi: D \to I$ such that $d_{i,\Phi(d_i)}=1$ for all $1 \leq i \leq m$, then *Paths1*(Φ, m, n, D, I) can produce m disjoint paths $Q_1, Q_2, \ldots, Q_m$ from s to $d_1, d_2, \ldots, d_m$, respectively, in an n-cube so that Q_i is shortest with length $|d_i|$ for all $1 \leq i \leq m$.

In [16], a one-to-one mapping $\Omega: \{d_1, d_2, \ldots, d_m\} \to \{1, 2, \ldots, n\}$ was referred to as a *partial routing function*, where $m \leq n$. Obviously, we can obtain a routing function Φ by defining $I=\{\Omega(d_1), \Omega(d_2), \ldots, \Omega(d_m)\}$ and $\Phi(d_i)=\Omega(d_i)$ for all $1 \leq i \leq m$, and we have $d_{i,\Phi(d_i)}=d_{i,\Omega(d_i)}=1$ if $d_{i,\Omega(d_i)}=1$. By the aid of Lemma 1, we have that if there exists a partial routing function Ω such that $d_{i,\Omega(d_i)}=1$ for all $1 \leq i \leq m$, then there exist m disjoint shortest paths from s to $d_1, d_2, \ldots, d_m$. In [16], a partial routing function Ω was said to be *maximal* if $\Sigma_\Omega \geq \Sigma_{\Omega'}$ for any $\Omega': \{d_1, d_2, \ldots, d_m\} \to \{1, 2, \ldots, n\}$, where $\Sigma_\Omega = \sum_{i=1}^{m} d_{i,\Omega(d_i)}$. A maximal Ω with $\Sigma_\Omega = m$ makes sure that $d_{i,\Omega(d_i)}=1$ for all $1 \leq i \leq m$, which implies the existence of m disjoint shortest paths from s to $d_1, d_2, \ldots, d_m$. It was shown in [16] that a maximal Ω can be computed in $O(mn^{1.5})$ time by solving a corresponding maximum bipartite matching problem.

3 Optimal disjoint paths in an *n*-fcube for odd *n*

Suppose that s is the source node and $t_1, t_2, \ldots, t_m$ are m (not necessarily distinct) target nodes in an n-fcube, where $m \leq n+1$, $s \notin \{t_1, t_2, \ldots, t_m\}$, and n is odd. Since folded hypercubes are node symmetric, we may assume $s=\overbrace{00\ldots0}^{n}=0^n$, i.e. the origin, without loss of generality. In this section, it will be shown that m disjoint paths from s to t_1, t_2, …, t_m, respectively, (in the n-fcube) can be constructed in $O(m^2n^{2.5})$ time so that not only is their total length minimized, but their maximal length is minimized in the worst case.

According to our construction method, each node t_i is first mapped to a node d_i in an $(n+1)$-cube for all $1\le i\le m$ by substituting the links of the $(n+1)$th dimension of $(n+1)$-cube for the complement links of n-fcube. The idea of node-mapping comes from the observation that any shortest path and nearly shortest path between two nodes in a folded hypercube contains at most one and two complement links, respectively. Then, by the aid of procedure *Paths1*, m disjoint paths Q_1, Q_2, …, Q_m from 0^{n+1} to d_1, d_2, …, d_m, respectively, can be constructed in the $(n+1)$-cube. As shown later, each path from s to t_i, denoted by R_i, can be derived from Q_i for all $1\le i\le m$ so that R_1, R_2, …, R_m are optimal and disjoint to each other. In the rest of this section, we first show how the node-mapping works as well as how Q_i and R_i are constructed. Then, we describe our main result formally.

Let V_n and V_{n+1} denote the two sets of nodes of an n-fcube (i.e. n-cube) and an $(n+1)$-cube, respectively. Since n is odd, we obtain the following node-mapping by slightly modifying that in [19].

Definition 1. Let $\varphi : V_n \to V_{n+1}$ be a node-mapping which maps a node $x= x_1x_2\bullet\bullet\bullet x_n\in V_n$ to a node $u = u_1u_2\bullet\bullet\bullet u_{n+1}\in V_{n+1}$ such that $u_{n+1}=0$ and $u_k=x_k$ for all $1\le k\le n$ if $|x|\le\lfloor n/2\rfloor$, $u_{n+1}=1$ and $u_j=1-x_k$ for all $1\le k\le n$ if $|x|>\lceil n/2\rceil$, and $u=\overbrace{11\ldots1}^{n+1}=1^{n+1}$ if $|x|=\lceil n/2\rceil$, where x_k, u_k, $u_{n+1}\in\{0, 1\}$ for all $1\le k\le n$.

Let $c_i=\varphi(t_i)$ for all $1\le i\le m$. It is not difficult to verify that e_k (resp. 1^n) can be included in a shortest path from s to t_i in an n-fcube if and only if $c_{i,k}=1$ (resp. $c_{i,n+1}=1$), where $1\le k\le n$. In addition, $|c_i|$ is the distance from s to t_i in the n-fcube if $|t_i|\ne\lceil n/2\rceil$. Suppose that $\Omega : \{c_1, c_2, …, c_m\} \to \{1, 2, …, n+1\}$ is a maximal partial routing function with $\Sigma_\Omega=r$, where $0\le r\le m$. Without loss of generality, we assume that $c_{i,\Omega(c_i)}=1$ for all $1\le i\le r$ and $c_{j,\Omega(c_j)}=0$ for all $r+1\le j\le m$. Moreover, we assume that there do not exist $r+1\le w\le m$ and $1\le z\le r$ such that $c_w\ne c_z$, $c_{w,k}\le c_{z,k}$ for all $1\le k\le n+1$, and there exists a partial routing function $\Omega' : \{c_1, c_2, …, c_{z-1}, c_w, c_{z+1}, …, c_r\} \to \{1, 2, …, n+1\}$ with $\Sigma_{\Omega'}=r$. According to [16], such an Ω can be determined in $O(m^2(n+1)^{2.5})=O(m^2n^{2.5})$ time.

Lemma 2. In an n-fcube, m disjoint paths R_1, R_2, …, R_m from s to t_1, t_2, …, t_m, respectively, can be constructed in $O(m^2n^{2.5})$ time so that each R_i is either shortest with length $dist_{f(n)}(s, t_i)$ if $c_{i,\Omega(c_i)}=1$, or nearly shortest with length $dist_{f(n)}(s, t_i)+2$ otherwise ($c_{i,\Omega(c_i)}=0$) for all $1\le i\le m$, where $dist_{f(n)}(s, t_i)$ is the distance from s to t_i in the n-fcube.

Proof. Without loss of generality, we may assume $|c_j|=\lfloor n/2\rfloor$ for all $r+1\le j\le\pi$, where $r\le\pi\le m$. For all $1\le i\le r$, let $d_i=c_i$ if $|t_i|\ne\lceil n/2\rceil$. When $|t_i|=\lceil n/2\rceil$, let $d_{i,n+1}=0$ and $d_{i,k}=t_{i,k}$ for all $1\le k\le n$ if $\Omega(c_i)\ne n+1$ and $t_{i,\Omega(c_i)}=1$, and let $d_{i,n+1}=1$ and $d_{i,k}=1-t_{i,k}$ for all $1\le k\le n$ otherwise (either $\Omega(c_i)=n+1$ or ($\Omega(c_i)\ne n+1$ and $t_{i,\Omega(c_i)}=0$)). For all $\pi+1\le j\le m$, let $d_j=c_j$ and $d'_j=d_{j,1}d_{j,2}\bullet\bullet\bullet d_{j,\Omega(c_j)-1}(1- d_{j,\Omega(c_j)})\, d_{j,\Omega(c_j)+1}\bullet\bullet\bullet d_{j,n+1}$.

For all $r+1\le j\le\pi$, let $t'_j\notin\{t_1, t_2, …, t_m, t'_{r+1}, …, t'_{j-1}\}$ such that $|t'_j|=\lceil n/2\rceil$ and t'_j is a neighbor of t_j in the n-fcube. We have $|t_w|\ne\lceil n/2\rceil$ for all $r+1\le w\le m$ because if $|t_z|=\lceil n/2\rceil$ for some $r+1\le z\le m$, then $c_z=\varphi(t_z)= 1^{n+1}$ and hence $c_{z,\Omega(c_z)}=1$, which contradicts to our assumptions. It follows that $t'_j\notin\{t_1, t_2, …, t_r, t'_{r+1}, …, t'_{j-1}\}$. Please refer to [20] for the existence of t'_j. When $\Omega(c_j)\ne n+1$ and $t'_{j,\Omega(c_j)}=1$, let $d'_{j,n+1}=0$ and $d'_{j,k}=t'_{j,k}$ for all $1\le k\le n$, and either let $d_{j,n+1}=0$ and $d_{j,k}=t_{j,k}$ if (t_j, t'_j) is not a complement link, or let $d_{j,n+1}=1$ and $d_{j,k}=1-t_{j,k}$ otherwise, for all $1\le k\le n$. On the other hand (either $\Omega(c_j)=n+1$ or ($\Omega(c_j)\ne n+1$ and $t'_{j,\Omega(c_j)}=0$)), let $d'_{j,n+1}=1$ and $d'_{j,k}=1-t'_{j,k}$ for all $1\le k\le n$, and either let $d_{j,n+1}=1$ and $d_{j,k}=1-t_{j,k}$ if (t_j, t'_j) is not a complement link, or let $d_{j,n+1}=0$ and $d_{j,k}=t_{j,k}$ otherwise, for all $1\le k\le n$. It is not difficult to verify that (d'_j, d_j) is a link in $(n+1)$-cube.

Define a routing function Φ: $\{d_1, d_2, …, d_r, d'_{r+1}, …, d'_m\} \to \{\Omega(c_1), \Omega(c_2), …, \Omega(c_m)\}$ such that $\Phi(d_i)=\Omega(c_i)$ for all $1\le i\le r$ and $\Phi(d'_j)=\Omega(c_j)$ for all $r+ 1\le j\le m$. It was shown in [20] that $d_{i,\Phi(d_i)}=1$ for all $1\le i\le r$ and $d'_{j,\Phi(d'_j)}=1$ for all $r+1\le j\le m$. By Lemma 1, *Paths1*(Φ, m, $n+1$, $\{d_1, d_2, …, d_r, d'_{r+1}, …, d'_m\}$, $\{\Omega(c_1), \Omega(c_2), …, \Omega(c_m)\}$) can produce m disjoint paths Q_1, Q_2, …, Q_r, Q'_{r+1}, …, Q'_m from 0^{n+1} to d_1, d_2, …, d_r, d'_{r+1}, …, d'_m, respectively, in an $(n+1)$-cube so that Q_i and Q'_j are both shortest with lengths $|d_i|$ and $|d'_j|$, respectively, for all $1\le i\le r$ and $r+1\le j\le m$. For all $r+1\le j\le m$, construct Q_j as the combination of Q'_j and a link (d'_j, d_j), and Q_j has length $|d'_j|+1$. By taking advantage of the properties of the preferred Ω (assumed above), it was shown in [20] that Q_1, Q_2, …, Q_m are disjoint to each other.

Define $\phi : V_{n+1} \to V_n$ as a node-mapping which maps a node $u =u_1u_2\bullet\bullet\bullet\, u_{n+1}\in V_{n+1}$ to a node $x = x_1x_2\bullet\bullet\bullet x_n\in V_n$ such that for all $1\le j\le n$, $x_j=u_j$ if $u_{n+1}=0$, and $x_j=1-u_j$ else ($u_{n+1}=1$). It is easy to verify that for every link (u, v) in an $(n+1)$-cube, there is a link $(\phi(u), \phi(v))$ in an n-fcube. For $1\le i\le m$, let R_i be constructed as follows: For each link (u, v) in Q_i, we have link $(\phi(u), \phi(v))$ included in R_i. By definition, we have either $d_{i,n+1}=0$ and $d_{i,k}=t_{i,k}$, or $d_{i,n+1}=1$ and $d_{i,k}=1-t_{i,k}$, for all $1\le k\le n$. It follows that $\phi(d_i)=t_i$, which implies that R_i is a path from s to t_i in an n-fcube because $\phi(0^{n+1})=s$ and Q_i is a path from 0^{n+1} to d_i in the $(n+1)$-cube. In addition, R_i and Q_i have the same length. Please refer to [20], R_1, R_2, …, R_m are disjoint to each other. For all $1\le i\le m$, each R_i is either shortest with length $dist_{f(n)}(s, t_i)$ if $c_{i,\Omega(c_i)}=1$, or nearly shortest with length $dist_{f(n)}(s, t_i)+2$ otherwise ($c_{i,\Omega(c_i)}=0$), as shown below. Since the length of R_i is $|d_i|$ for all $1\le i\le r$ and the length of R_j is $|d'_j|+1$ for all $r+1 \le j \le m$, it is sufficient to show $|d_i|=dist_{f(n)}(s, t_i)$ and $|d'_j|+1=dist_{f(n)}(s, t_j)+2$. For all $1 \le i \le r$, if $|t_i| \ne \lceil n/2 \rceil$, then we have $d_i=c_i$, which implies $|d_i|=|c_i|=dist_{f(n)}(s, t_i)$. Otherwise ($|t_i|=\lceil n/2 \rceil$), we have either $d_{i,n+1}=0$ and $d_{i,k}=t_{i,k}$, or $d_{i,n+1}=1$ and $d_{i,k}=1-t_{i,k}$, for all $1 \le k \le n$. It follows that either $|d_i|=|t_i|=\lceil n/2 \rceil$ or $|d_i|=(n+1)-|t_i|=(n+1)-\lceil n/2 \rceil=\lceil n/2 \rceil$, which implies $|d_i|=\lceil n/2 \rceil=dist_{f(n)}(s, t_i)$. For all $\pi+1 \le j \le m$, we have $d_j=c_j$ and $d'_j=d_{j,1}d_{j,2} \bullet\bullet\bullet d_{j,\Omega(c_j)-1}(1-d_{j,\Omega(c_j)}) d_{j,\Omega(c_j)+1} \bullet\bullet\bullet d_{j,n+1}$. Since $d_{j,\Omega(c_j)}=c_{j,\Omega(c_j)}=0$, we have $|d'_j|=|c_j|+1$, which implies $|d'_j|+1=|c_j|+2=dist_{f(n)}(s, t_j)+2$. For all $r+1 \le j \le \pi$, we have either $d'_{j,n+1}=0$ and $d'_{j,k}=t'_{j,k}$, or $d'_{j,n+1}=1$ and $d'_{j,k}=1-t'_{j,k}$, for all $1 \le k \le n$. Since $|t'_j|=\lceil n/2 \rceil$, we have either $|d'_j|=|t'_j|=\lceil n/2 \rceil$ or $|d'_j|=(n+1)-|t'_j|=(n+1)-\lceil n/2 \rceil=\lceil n/2 \rceil$, which implies $|d'_j|=\lceil n/2 \rceil$. Hence, $|d'_j|+1=\lceil n/2 \rceil+1=\lfloor n/2 \rfloor+2=|c_j|+2=dist_{f(n)}(s, t_j)+2$.

It is not difficult to verify that all of $d_1, d_2, \ldots, d_m, d'_{\pi+1}, \ldots, d'_m$ can be obtained in $O(m)$ time, all of $d'_{r+1}, \ldots, d'_\pi$ can be obtained in $O(mn^2)$ time, and each R_i can be derived from Q_i in $O(n)$ time for all $1 \le i \le m$. Since procedure *Paths1* takes $O(mn)$ time and the preferred Ω (assumed above) can be computed in $O(m^2n^{2.5})$ time, the construction of $R_1, R_2, \ldots, R_m$ takes $O(m^2n^{2.5})$ time as a whole. □

If $|t_i|=\lceil n/2 \rceil$, then we have $c_i=\varphi(t_i)=1^{n+1}$, which implies $c_{i,\Omega(c_i)}=1$. Lemma 2 ensures that the length of R_i is $\lceil n/2 \rceil$. On the other hand ($|t_i| \ne \lceil n/2 \rceil$), Lemma 2 ensures that the length of R_i is at most $dist_{f(n)}(s, t_i)+2 \le \lfloor n/2 \rfloor+2=\lceil n/2 \rceil+1$. Since there exists a case in [4] such that its maximal length is not less than $\lceil n/2 \rceil+1$, the maximal length of $R_1, R_2, \ldots, R_m$ is minimized in the worst case. The total length of $R_1, R_2, \ldots, R_m$ is minimized, as shown below. Since an n-fcube is bipartite for odd n (see [29]), the lengths of all the paths from a node to another have the same parity, and hence the length of a nearly shortest (or second shortest) path must equal to the distance (between the two end nodes) plus two. If there exist m disjoint paths $R'_1, R'_2, \ldots, R'_m$ from s to $t_1, t_2, \ldots, t_m$, respectively, so that their total length is less than that of $R_1, R_2, \ldots, R_m$, then we have $r'>r$ shortest paths in $R'_1, R'_2, \ldots, R'_m$. Define a partial routing function $\Omega' : \{c_1, c_2, \ldots, c_m\} \rightarrow \{1, 2, \ldots, n+1\}$ as follows: $\Omega'(c_i)=k_i$ (resp. $\Omega'(c_i)= n+1$) if e_{k_i} (resp. 1^n) is included R'_i all $1 \le i \le m$. If R'_w is shortest, then we have either $c_{w,\Omega'(c_w)}=c_{w,k_w}=1$ or $c_{w,\Omega'(c_w)}=c_{w,n+1}=1$, where $1 \le w \le m$. It follows that we have $\Sigma_{\Omega}' \ge r' > r$, which contradicts to that Ω is maximal. Since folded hypercubes are node- symmetric, we have the following theorem, which is the main result of this paper.

Theorem 1. In an n-dimensional folded hypercube, there exist m node-disjoint paths from one source node to other m (not necessarily distinct) target nodes so that not only is their total length minimized, but their maximal length is also minimized in the worst case, where $m \le n+1$ and n is odd. In addition, each path is either shortest or nearly shortest with length not greater than the minimum of $\lceil n/2 \rceil+1$ and the distance (from source node to target node) plus two, where $\lceil n/2 \rceil$ is the diameter of the n-dimensional folded hypercube.

4 Concluding remarks and future works

In this paper, we focus on the problem of constructing m disjoint paths $R_1, R_2, \ldots, R_m$ from one source node s to other m (not necessarily distinct) target nodes $t_1, t_2, \ldots, t_m$, respectively, in an n-fcube so that not only is their total length minimized, but their maximal length is minimized in the worst case, where $m \le n+1$ and n is odd. In addition, such $R_1, R_2, \ldots, R_m$ can be constructed in $O(m^2n^{2.5})$ time, which is dominated by the computation of the preferred maximal partial routing function Ω. Since a maximal partial routing function can be computed in just $O(mn^{1.5})$ time, we conjecture that the total time complexity can be further reduced if the construction method can be slightly modified. On the other hand, when n is even, the n-fcube is no longer bipartite, which makes the construction of $R_1, R_2, \ldots, R_m$ more difficult. Another kind of routing functions may be needed to make their total length minimized for even n, and hence it is currently under our study [20].

References

[1] C. C. Chen and J. Chen, "Nearly optimal one-to- many parallel routing in star networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 8, no. 12, pp. 1196-1202, 1997.

[2] X. B. Chen, "Many-to-many disjoint paths in faulty hypercubes," *Information Sciences*, vol. 179, no. 18, pp. 3110-3115, 2009.

[3] M. Dietzfelbinger, S. Madhavapeddy, and I. H. Sudborough, "Three disjoint path paradigms in star networks," *Proceedings of the third IEEE Symposium on Parallel and Distributed Processing*, 1991, pp. 400-406.

[4] D. R. Duh and G. H. Chen, "On the Rabin number problem," *Networks*, vol. 30, no. 3, pp. 219-230, 1997.

[5] A. El-Amawy and S. Latifi, "Properties and performance of folded hypercubes," *IEEE Transactions on Parallel and Distributed Systems*, vol. 2, no. 1, pp. 31-42, 1991.

[6] J. F. Fang, "The bipanconnectivity and *m*-panconnectivity of the folded hypercube," *Theoretical Computer Science*, vol. 385, no. 1-3, pp. 286-300, 2007.

[7] J. S. Fu, "Fault-free cycles in folded hypercubes with more faulty elements," *Information Processing Letters*, vol. 108, no. 5, pp. 261-263, 2008.

[8] Z. Galil, "Efficient algorithms for finding maximum matching in graphs," *ACM Computing Surveys*, vol. 18, no. 1, pp. 23-38, 1986.

[9] S. Gao and D. F. Hsu, "Short containers in Cayley graphs," *Discrete Applied Mathematics*, vol. 157, pp. 1354-1363, 2009.

[10] Q. P. Gu and S. Peng, "An efficient algorithm for the *k*-pairwise disjoint paths problem in hypercubes,"

Journal of Parallel and Dis- tributed Computing, vol. 60, pp. 764-774, 2000.

[11] Q. P. Gu and S. Peng, "Node-to-set and set-to- set cluster fault tolerant routing in hypercubes," *Parallel Computing*, vol. 24, pp. 1245-1261, 1998.

[12] S. Y. Hsieh, "Some edge-fault-tolerant properties of the folded hypercube", *Networks*, vol. 51, no. 2, pp.92-101, 2007.

[13] S. Y. Hsieh, C. Y. Tsai, and C. A. Chen, "Strong diagnosability and conditional diagnosability of multiprocessor systems and folded hypercubes," *IEEE Transactions on Computers*, vol. 62, no. 7, pp. 1472-1477, 2013.

[14] C. N. Kuo and Y. H. Cheng, "Cycles embedding in folded hypercubes with conditionally faulty vertices," *Discrete Applied Mathematics*, vol. 220, pp. 55-59, 2017.

[15] C. N. Kuo and S. Y. Hsieh, "Pancyclicity and bipancyclicity of conditional faulty folded hypercubes," *Information Sciences*, vol. 180, no. 15, pp. 2904-2914, 2010.

[16] C. N. Lai, "Optimal construction of all shortest node-disjoint paths in hypercubes with applications," *IEEE Transactions on Parallel and Distributed Systems*, vol. 23, no. 6, pp. 1129-1134, 2012.

[17] C. N. Lai, "Constructing all shortest node-disjoint paths in torus networks," *Journal of Parallel and Distributed Computing*, vol. 75, pp. 123-132, January 2015.

[18] C. N. Lai, "On the construction of all shortest vertex-disjoint paths in Cayley graphs of abelian groups," *Theoretical Computer Science*, vol. 571, pp. 10-20, March 2015.

[19] C. N. Lai, "Optimal construction of node-disjoint shortest paths in folded hypercubes," *Journal of Parallel and Distributed Computing*, vol. 102, pp. 37-41, April 2017.

[20] C. N. Lai, "Constructing optimal node-disjoint paths in folded hypercubes," manuscript, 2018.

[21] S. C. Liaw and G. J. Chang, "Generalized diameters and Rabin numbers of networks," *Journal of Combinatorial Optimization*, vol. 4, pp. 371-384, 1999.

[22] M. Ma, "The spanning connectivity of folded hypercubes," *Information Sciences*, vol. 180, no. 17, pp. 3373-3379, 2010.

[23] Y. Qiao and W. Yang, "Edge disjoint paths in hypercubes and folded hypercubes with conditional faults," *Applied Mathematics and Computation*, vol. 294, pp. 96-101, 2017.

[24] M. O. Rabin, "Efficient dispersal of information for security, load balancing, and fault tolerance," *Journal of the ACM*, vol. 36, no. 2, pp. 335-348, 1989.

[25] Y. Saad and M. H. Schultz, "Topological properties of hypercubes," *IEEE Transactions on Computers*, vol. 37, no. 7, pp. 867-872, 1988.

[26] D. Wang, "Embedding Hamiltonian cycles into folded hypercubes with faulty links," *Journal of Parallel and Distributed Computing*, vol. 61, pp. 545-564, 2001.

[27] R. Y. Wu, G. H. Chen, Y. L. Kuo, and G. J. Chang, "Node-disjoint paths in hierarchical hypercube networks," *Information Sciences*, vol. 177, no. 19, pp. 4200-4207, 2007.

[28] Y. Xiang and I. A. Stewart, "One-to-many node-disjoint paths in (*n*, *k*)-star graphs," *Discrete Applied Mathematics*, vol. 158, pp. 62-70, 2010.

[29] J. M. Xu and M. Ma, "Cycles in folded hypercubes," *Applied Mathematics Letters*, vol. 19, pp. 140-145, 2006.

[30] W. Yang, S. Zhao, and S. Zhang, "Strong Menger connectivity with conditional faults of folded hypercubes," *Information Processing Letters*, vol. 125, pp. 30-34, 2017.

[31] Q. Zhu, J. M. Xu, X. Hou, and M. Xu, "On reliability of the folded hypercubes," *Information Sciences*, vol. 177, pp. 1782-1788, 2007.