The Weakly Dimension-Balanced Hamiltonian

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Abstract—Given a graph \( G = (V, E) \) and a cycle \( C \) on \( G \), the edge set \( E \) is partitioned into \( k \) dimensions for a positive integer \( k \). The set of all \( i \)-dimensional edge of \( C \), a subset of \( E(C) \), is denoted as \( E_i(C) \) for \( 1 \leq i \leq k \). If \( |E_i(C)| - |E_j(C)| \leq 1 \) for \( 1 \leq i < j \leq k \), \( C \) is called a dimension-balanced cycle. If \( C \) is also a Hamiltonian cycle of \( G \), \( C \) is called a dimension-balanced Hamiltonian cycle. Furthermore, if \( |E_i(C)| - |E_j(C)| \leq 3 \) for \( 1 \leq i < j \leq k \), \( C \) is called a weakly dimension-balanced cycle. If \( C \) is also a Hamiltonian cycle of \( G \), \( C \) is called a weakly dimension-balanced Hamiltonian cycle of \( G \) which is weakly DB Hamiltonian. In this paper, we prove that for any \( m, n \geq 3 \), a toroidal mesh graph \( T_{m,n} \) is weakly DB Hamiltonian.

Keywords—Hamiltonian, Toroidal mesh graph, dimension-balanced, weakly dimension-balanced cycle

I. INTRODUCTION

It is well-known that an interconnection network is usually represented by a graph where vertices represent processors, and edges represent communication links between processors. For a graph \( G = (V, E) \), where \( V(G) \) is the vertex set and \( E(G) \) is the edge set, \( |V(G)| \) denotes the number of vertex set and \( |E(G)| \) denotes the number of edge set. Hamiltonicity is an important property and has been widely discussed in the literature (see [1, 2, 3, 4, 5, 6, 7, 8]). A Hamiltonian cycle of \( G \) is a cycle that contains every vertex of \( G \). \( T_{m,n} \) is called the toroidal mesh graph whose vertex set \( V(T_{m,n}) = \{ (x, y) | 0 \leq x \leq m - 1, 0 \leq y \leq n - 1 \} \), and edge set \( E(T_{m,n}) = \{ (x_1, y_1) (x_2, y_2); x_1 = x_2 \text{ and } y_1 - y_2 = \pm 1 \text{ (mod } m) \text{, or } y_1 = y_2 \text{ and } x_1 - x_2 = \pm 1 \text{ (mod } m) \} \). Fig. 1 shows an example of \( T_{3,4} \).

Throughout this paper, for the graph theoretical definitions and notations we follow [9]. Given a graph \( G = (V, E) \), the edge set \( E \) is partitioned into \( k \) dimensions \( E_1, E_2, \ldots, E_k \) for a positive integer \( k \). For any cycle \( C \) on \( G \), the set of all \( i \)-dimensional edge of \( C \), which is a subset of \( E(C) \), is denoted as \( E_i(C) \). If \( |E_i(C)| - |E_j(C)| \leq 1 \) for \( 1 \leq i < j \leq k \), \( C \) is called a dimension-balanced cycle. If \( C \) is also a Hamiltonian cycle of \( G \), \( C \) is called a dimension-balanced Hamiltonian cycle. Furthermore, if \( |E_i(C)| - |E_j(C)| \leq 3 \) for \( 1 \leq i < j \leq k \), \( C \) is called a weakly dimension-balanced cycle. If \( C \) is also a Hamiltonian cycle of \( G \), \( C \) is called a weakly dimension-balanced Hamiltonian cycle of \( G \) which is weakly DB Hamiltonian. In this paper, we prove that for any \( m, n \geq 3 \), a toroidal mesh graph \( T_{m,n} \) is weakly DB Hamiltonian.

II. PRELIMINARY

Theorem 1. [11] For even m and n, $T_{m,n}$ is DB Hamiltonian.

Theorem 2. [11] If one of m, n is a multiple of 4, the other is odd, $T_{m,n}$ is DB Hamiltonian.

Theorem 3. [11] $T_{m,n}$ is not DB Hamiltonian for $mn \mod 4 = 2$.

Theorem 4. [10] For m is odd, $n \mod 4 = 1$ and m, n $\geq 3$, $T_{m,n}$ is DB Hamiltonian.

Theorem 5. [10] For $m \geq 5$ is odd, $n \mod 4 = 3$ and $n \geq 7$, $T_{m,n}$ is DB Hamiltonian.

Lemma 1. [10] For $n = 3$, and m $\geq 3$ is odd, $T_{m,n}$ is DB Hamiltonian.

Corollary 1. [10] For m, n $\geq 3$, there is a Hamiltonian DBC on $T_{m,n}$, except for the state on mn mod 4 = 2.

By definition, a DBC is also a WDBC. According to previous research, the current results on the toroidal mesh graph $T_{m,n}$ are summarized in Table 1. We list the existence of a weakly dimension-balanced Hamiltonian cycle in different m and n on a $T_{m,n}$. The unknown part is discussed in next Section.

Table 1: The existence of a Hamiltonian WDBC on $T_{m,n}$ for m, n $\geq 3$

<table>
<thead>
<tr>
<th>m mod 4 = 0</th>
<th>m mod 4 = 2</th>
<th>m is odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>n mod 4 = 0</td>
<td>Yes (Theorem 1)</td>
<td>Yes (Theorem 2)</td>
</tr>
<tr>
<td>n mod 4 = 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>n mod 4 = 1</td>
<td>Yes (Theorem 3)</td>
<td></td>
</tr>
<tr>
<td>n mod 4 = 3, n $\geq 7$</td>
<td>Yes (Theorem 4, Lemma 1)</td>
<td></td>
</tr>
<tr>
<td>n = 3</td>
<td>Yes (Lemma 1)</td>
<td></td>
</tr>
</tbody>
</table>

III. MAIN RESULTS

In this section, we prove that there exists a weakly dimension-balanced Hamiltonian cycle on $T_{m,n}$ for positive integers m, n $\geq 3$ with mn mod 4 = 2 by the following theorem. Then, we will conclude the main result.

Theorem 6. For m, n $\geq 3$ and mn mod 4 = 2, there is a Hamiltonian WDBC on $T_{m,n}$.

Proof. When mn mod 4 = 2, one of the following situations will hold. (i) m mod 4 = 2 and n is odd; (ii) n mod 4 = 2 and m is odd. The proof for both are similar, so we discuss (i) m mod 4 = 2 and n is odd in follows. The proof is divided into two cases. Case 1 discusses the condition on m mod 4 = 2 and n = 3 + 4y for a nonnegative integer y; Case 2 discusses the state of m mod 4 = 2 and n = 5 + 4y for a nonnegative integer y.

Case 1: m mod 4 = 2, n = 3 + 4y for nonnegative integer y

In this case, we consider m = 6 and n = 3 + 4y for a nonnegative integer y. We construct a Hamiltonian WDBC $W_{6,3}$ of $T_{6,3}$ and a Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$ as Fig. 2. Then, Fig. 3 shows how to apply these two cycles to form a Hamiltonian WDB of $T_{m,n}$ for $n = 3 + 4y$ and y is a nonnegative integer: stack $H_{6,4}$ y times on the top of $W_{6,3}$. Then, delete edge set $E_1 = \{(0, 3 + 4i)(1, 3 + 4i), (0, 6 + 4i)(1, 6 + 4i) | 0 \leq i \leq y - 1\} \cup \{(0, 0)(2, 0), (1, 0)(1, 2)\}$, and add edge set $E_2 = \{(0, 2 + 4i)(0, 3 + 4i)(1, 2 + 4i)(1, 4 + 4i) | 0 \leq i \leq y - 1\} \cup \{(0, 0)(0, n - 1), (1, 0)(1, n - 1)\}$.

After these steps, a Hamiltonian cycle $W_{6,3}$ on $T_{6,3}$ is constructed, where $|E_1(W_{6,3})| = 10 + (14 - 2)y = 10 + 12y$ and $|E_2(W_{6,3})| = 8 + (10 + 2)y = 8 + 12y$. Consequently, $|E_1(W_{6,3})| - |E_2(W_{6,3})| = 2$, $W_{6,3}$ satisfies the definition of Hamiltonian WDBC.

Fig. 2: (a) A Hamiltonian WDBC $W_{6,3}$ on $T_{6,3}$. (b) A Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$

Fig. 3: A Hamiltonian WDBC $W_{6,3}$ on $T_{6,3}$
For $m > 6$, we want to construct a Hamiltonian DBC for $T_{4,n}$. At first, we give a Hamiltonian DBC $B_{4,3}$ of $T_{4,3}$ and a Hamiltonian cycle $H_{4,4}$ on $T_{4,4}$ as Fig. 4. Besides, Fig. 5 shows how to use these two cycles to form a Hamiltonian WDB of $T_{4,n}$ for $n = 3 + 4y$ and $y$ is a nonnegative integer: stack $H_{4,4}$ $y$ times on the top of $B_{4,3}$. Then, delete edge set $E_3 = \{(2, 3 + 4i)(3, 3 + 4i), (2, 6 + 4i)(3, 6 + 4i) | 0 \leq i \leq y - 1\} \cup \{(2, 0)(2, 2), (3, 0)(3, 2)\}$, and add edge set $E_4 = \{(2, 2 + 4i)(2, 3 + 4i), (3, 2 + 4i)(3, 3 + 4i) | 0 \leq i \leq y - 1\} \cup \{(2, 0)(2, n - 1), (3, 0)(3, n - 1)\}$. After these steps, a Hamiltonian cycle $B_{4, n}$ is generated, where $|E_1(B_{4, n})| = 6 + (10 - 2)y = 6 + 8y$ and $|E_2(B_{4, n})| = 6 + (6 + 2)y = 6 + 8y$. Consequently, $|E_1(B_{4, n})| - |E_2(B_{4, n})| = 0$, $B_{4, n}$ satisfies the definition of the Hamiltonian DBC.

Fig. 5: A Hamiltonian DBC $B_{4, n}$ of $T_{4, n}$

Finally, let $x = (m - 6) / 4$, put $B_{4, n}$ for $x$ times on right side of $W_{6,n}$. Then, delete edge set $E_5 = \{(6 + 4i, 2)(9 + 4i, 2) | 0 \leq i \leq x - 1\} \cup \{(0, 2)(5, 2)\}$, and add edge set $E_6 = \{(5 + 4i, 2)(6 + 4i, 2) | 0 \leq i \leq x - 1\} \cup \{(0, 2)(m - 1, 2)\}$, show in Fig. 6. After these steps, a Hamiltonian cycle $W_{6,n}$ on $T_{m,n}$ for $m \mod 4 = 2$ and $n = 3 + 4y$ for a nonnegative integer $y$ is constructed, where $|E_1(W_{6,n})| = (10 + 12y) + (6 + 8y)x = 8xy + 6x + 12y + 10$ and $|E_2(W_{6,n})| = (8 + 12y) + (6 + 8y)x = 8xy + 6x + 12y + 8$. Consequently, $|E_1(W_{6,n})| - |E_2(W_{6,n})| = 2$, $W_{6,n}$ satisfies the definition of weakly dimension-balanced Hamiltonian cycle.

Case 2. $m \mod 4 = 2$, $n = 5 + 4y$ for nonnegative integer $y$

In this case, at first, we consider $m = 6$ and $n = 5 + 4y$ for a nonnegative integer $y$. We construct a WDBC $W_{6,5}$ on $T_{5,n}$ and a Hamiltonian cycle $H_{6,4}$ on $T_{5,4}$ as shown in Fig. 7. Besides, Fig. 8 shows how to apply these two cycles to construct a Hamiltonian WDB of $T_{6,n}$ for $n = 5 + 4y$ and $y$ is a nonnegative integer: stack $H_{6,4}$ $y$ times on the top of $W_{6,5}$. Then, delete edge set $E_7 = \{(0, 5 + 4i)(1, 5 + 4i), (0, 8 + 4i)(1, 8 + 4i) | 0 \leq i \leq y - 1\} \cup \{(0, 0)(0, 4), (1, 0)(1, 4)\}$, and add edge set $E_8 = \{(0, 4 + 4i)(0, 5 + 4i), (1, 4 + 4i)(1, 5 + 4i) | 0 \leq i \leq y - 1\} \cup \{(0, 0)(n - 1), (1, 0)(n - 1)\}$. After these steps, a Hamiltonian cycle $W_{6, n}$ on $T_{5,n}$ is constructed, where $|E_1(W_{6,n})| = 16 + (14 - 2)y = 16 + 12y$ and $|E_2(W_{6,n})| = 14 + (10 + 2)y = 14 + 12y$. Consequently, $|E_1(W_{6,n})| - |E_2(W_{6,n})| = 2$, $W_{6,n}$ satisfies the definition of weakly dimension-balanced Hamiltonian cycle.

Fig. 6: A Hamiltonian WDBC on $T_{m,n}$ for $m \mod 4 = 2$ and $n = 3 + 4y$ for nonnegative integer $y$

Fig. 7: (a) A Hamiltonian WDBC $W_{6,5}$ on $T_{6,5}$, (b) A Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$

When $m > 6$, we construct a Hamiltonian DBC for $T_{4,n}$. At first, we give a Hamiltonian DBC $B_{4,5}$ of $T_{4,5}$ and a
Hamiltonian cycle \( H_{4,4} \) on \( T_{4,4} \) as Fig. 9. Besides, Fig. 10 shows how to use these two cycles to form a Hamiltonian WDB of \( T_{a,n} \) for \( n = 5 + 4y \) and \( y \) is a nonnegative integer: stack \( H_{4,4} \) \( y \) times on the top of \( B_{4,5} \). Then, delete edge set \( \mathcal{E}_9 = \{(2, 5 + 4i)(3, 5 + 4i), (2, 8 + 4i)(3, 8 + 4i) \mid 0 \leq i \leq y - 1\} \cup \{(2, 0)(2, 4), (3, 0)(3, 4)\} \), and add edge set \( \mathcal{E}_{10} = \{(2, 4 + 4i)(2, 5 + 4i), (3, 4 + 4i)(3, 5 + 4i) \mid 0 \leq i \leq y - 1\} \cup \{(2, 0)(2, n - 1), (3, 0)(3, n - 1)\} \). After these steps, a Hamiltonian cycle \( B_{4,5} \) of \( T_{a,n} \) is generated, whose \(|\mathcal{E}(B_{4,5})| = 10 + (10 - 2)y = 10 + 8y \) and \( |\mathcal{E}_d(B_{4,5})| = 10 + (6 + 2)y = 10 + 8y \). Consequently, \(|\mathcal{E}(B_{4,5})| - |\mathcal{E}_d(B_{4,5})| = 0, \) \( B_{4,5} \) satisfies the definition of the Hamiltonian DBC.

Finally, let \( x = (m - 6)/4 \), put \( B_{4,5} \) for \( x \) times on right side of \( W_{m,n} \). Then, delete edge set \( \mathcal{E}_{11} = \{(6 + 4i, 4)(9 + 4i, 4) \mid 0 \leq i \leq x - 1\} \cup \{(0, 4)(5, 4)\} \), and add edge set \( \mathcal{E}_{12} = \{(5 + 4i, 4)(6 + 4i, 4) \mid 0 \leq i \leq x - 1\} \cup \{(0, 4)(m - 1, 4)\} \), as shown in Fig. 11. After these steps, a Hamiltonian cycle \( W_{m,n} \) on \( T_{m,n} \) for \( m \text{ mod } 4 = 2 \) and \( n = 5 + 4y \) for a nonnegative integer \( y \) is generated, whose \(|\mathcal{E}(W_{m,n})| = (16 + 12y) + (10 + 8y)x = 8xy + 10x + 12y + 16 \) and \( |\mathcal{E}_d(W_{m,n})| = (14 + 12y) + (10 + 8y)x = 8xy + 10x + 12y + 14 \). Consequently, \(|\mathcal{E}(W_{m,n})| - |\mathcal{E}_d(W_{m,n})| = 2, \) \( W_{m,n} \) satisfies the definition of weakly dimension-balanced Hamiltonian cycle.

By Theorem 6 and Corollary 1, Corollary 2 is the main result of this paper.

**Corollary 2.** For any \( m, n \geq 3 \), toroidal mesh graph \( T_{m,n} \) is weakly DB Hamiltonian.
IV. CONCLUSIONS

In this paper, we proved that the toroidal mesh graph $T_{m,n}$ contains a weakly dimension-balanced Hamiltonian cycle, where $m, n \geq 3$ and $mn \mod 4 = 2$. Referring to previous research works, we have concluded that $T_{m,n}$ contains a Hamiltonian WDBC for any $m, n \geq 3$. Thus, $T_{m,n}$ is weakly DB Hamiltonian.

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REFERENCES


