# The Weakly Dimension-Balanced Hamiltonian

**Ruei-Yu Wu[1], Zong-You Lai[2] and Justie Su-Tzu Juan[2, *]**

[1] Department of Management Information Systems,

Hwa Hsia University of Technology, New Taipei City, Taiwan.

Email: fish@cc.hwh.edu.tw

[2] Department of Computer Science and Information Engineering,

National Chi Nan University, Puli, Nantou, Taiwan.

Email: s103321003@mail1.ncnu.edu.tw

[*]Correponding author: jsjuan@ncnu.edu.tw

***Abstract*—Given a graph $G = (V, E)$ and a cycle $C$ on $G$, the edge set $E$ is partition into $k$ dimensions for a positive integer $k$. The set of all $i$-dimensional edge of $C$, a subset of $E(C)$, is denoted as $E_i(C)$ for $1 \le i \le k$. If $||E_i(C)| - |E_j(C)|| \le 1$ for $1 \le i < j \le k$, $C$ is called a dimension-balanced cycle. If $C$ is also a Hamiltonian cycle of $G$, $C$ is called a dimension-balanced Hamiltonian cycle. Furthermore, if $||E_i(C)| - |E_j(C)|| \le 3$ for $1 \le i < j \le k$, $C$ is called a weakly dimension-balanced cycle. If $C$ is also a Hamiltonian cycle of $G$, $C$ is called a weakly dimension-balanced Hamiltonian cycle of $G$ which is weakly DB Hamiltonian. In this paper, we prove that for any $m$, $n \ge 3$, a toroidal mesh graph $T_{m,n}$ is weakly DB Hamiltonian.**

***Keywords*—Hamiltonian, Toroidal mesh graph, dimension-balanced, weekly dimension-balanced cycle**

## I. Introduction

It is well-know that an interconnection network is usually represented by a graph where vertices represent processors, and edges represent communication links between processors. For a graph $G = (V, E)$ , where $V(G)$ is the *vertex set* and $E(G)$ is the *edge set*, $|V(G)|$ denotes the number of vertex set and $|E(G)|$ denotes the number of edge set. Hamiltonicity is an important property and has been widely discussed in the literature (see [1, 2, 3, 4, 5, 6, 7, 8]). A *Hamiltonian cycle* of $G$ is a cycle that contains every vertex of $G$. $T_{m,n}$ is called the *toroidal mesh graph* whose vertex set $V(T_{m,n}) = \{(x, y)| 0 \le x \le m - 1, 0 \le y \le n - 1\}$, and edge set $E(T_{m,n}) = \{(x_1, y_1)\ (x_2, y_2)|\ x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod n$, or $y_1 = y_2$ and $x_1 - x_2 \equiv \pm 1 \pmod m\}$. Fig.1 shows an example of $T_{3,4}$.

Throughout this paper, for the graph theoretical definitions and notations we follow [9]. Given a graph $G = (V, E)$, the edge set $E$ be partitioned into $k$ dimensions $E_1, E_2, \ldots, E_k$ for a positive integer $k$. For any cycle $C$ on $G$, the set of all $i$-dimensional edge of $C$, which is a subset of $E(C)$, is denoted as $E_i(C)$. If $||E_i(C)| - |E_j(C)|| \le 1$ for any $1 \le i < j \le k$, $C$ is called a *dimension-balanced cycle* (DBC, for short). Let $C$ be a Hamiltonian cycle ($C$ contains every vertex of graph $G$ exactly once.) on graph $G$, if $C$ is a DBC, $C$ is called a *dimension-balanced Hamiltonian cycle* (Hamiltonian DBC, for short). If there exists a Hamiltonian DBC in a graph $G$, $G$ is called *DB Hamiltonian*. When a graph has no dimension-balanced Hamiltonian cycle, we try to check whether $G$ meets some criteria. Then, we extend the definition as follows. If $||E_i(C)| - |E_j(C)|| \le 3$ for any $1 \le i < j \le k$, a cycle $C$ is called a *weakly dimension-balanced cycle* (WDBC, for short). Let $C$ be a WDBC on $G$, if $C$ is also a Hamiltonian cycle, $C$ is called a *weakly dimension-balanced Hamiltonian cycle* (Hamiltonian WDBC, for short). If there exists a Hamiltonian WDBC in $G$, $G$ is called *weakly DB Hamiltonian*. In this paper, we prove that the toroidal mesh graph $T_{m,n}$ contains a weakly dimension-balanced Hamiltonian cycle for any $m$, $n \ge 3$. Section 2 discusses some preliminary of the main problem. In Section 3, we show the proof for the main result. Finally, this paper concludes in Section 4.

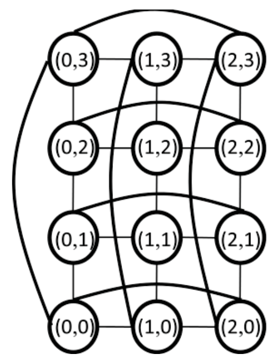


Fig 1: The structure of $T_{3,4}$.

## II. Preliminary

**Theorem 1.** [11] *For even* $m$ *and* $n$, $T_{m,n}$ *is DB Hamiltonian*.

**Theorem 2.** [11] *If one of* $m$, $n$ *is a multiple of* 4, *the other is odd*, $T_{m,n}$ *is DB Hamiltonian*.

**Theorem 3.** [11] $T_{m,n}$ *is not DB Hamiltonian for mn mod* $4 = 2$.

**Theorem 4**. [10] *For m is odd*, *n mod* $4 = 1$ *and* $m, n \geq 3$, $T_{m,n}$ *is DB Hamiltonian*.

**Theorem 5**. [10] *For* $m \geq 5$ *is odd*, *n mod* $4 = 3$ *and* $n \geq 7$, $T_{m,n}$ *is DB Hamiltonian*.

**Lemma 1.** [10] *For* $n = 3$, *and* $m \geq 3$ *is odd*, $T_{m,n}$ *is DB Hamiltonian*.

**Corollary 1.** [10] *For* $m, n \geq 3$, *there is a Hamiltonian DBC on* $T_{m,n}$, *except for the state on mn mod* $4 = 2$.

By definition, a DBC is also a WDBC. According to previous research, the current results on the toroidal mesh graph $T_{m,n}$ are summarized in Table 1. We list the existence of a weakly dimension-balanced Hamiltonian cycle in different $m$ and $n$ on a $T_{m,n}$. The unknown part is discussed in next Section.

Table 1: The existence of a Hamiltonian WDBC on $T_{m,n}$ for $m, n \geq 3$

| | $m$ mod 4 = 0 | $m$ mod 4 = 2 | $m$ is odd |
|---|---|---|---|
| $n$ mod 4 = 0 | Yes (Theorem 1) | | Yes (Theorem 2) |
| $n$ mod 4 = 2 | | | ? |
| $n$ mod 4 = 1 | Yes (Theorem 2) | ? | Yes (Theorem 3) |
| $n$ mod 4 = 3, $n \geq 7$ | | | Yes (Theorem 4 Lemma 1) |
| $n = 3$ | | | Yes (Lemma 1) |

## III. Main results

In this section, we prove that there exists a weakly dimension-balanced Hamiltonian cycle on $T_{m,n}$ for positive integers $m, n \geq 3$ with $mn$ mod $4 = 2$ by the following theorem. Then, we will conclude the main result.

**Theorem 6.** *For* $m, n \geq 3$ *and mn mod* $4 = 2$, *there is a Hamiltonian WDBC on* $T_{m,n}$.

**Proof.** When $mn$ mod $4 = 2$, one of the following situations will hold. (i) $m$ mod $4 = 2$ and $n$ *is odd*; (ii) $n$ mod $4 = 2$ and $m$ is odd. The proof for both are similar, so we discuss (i) $m$ mod $4 = 2$ and $n$ is odd in follows. The proof is divided into two cases. Case 1 discusses the condition on $m$ mod $4 = 2$ and $n = 3 + 4y$ for a nonnegative integer $y$; Case 2 discusses the state of $m$ mod $4 = 2$ and $n = 5 + 4y$ for a nonnegative integer $y$.

**Case 1 : *m* mod 4 = 2, *n* = 3 + 4*y* for nonnegative integer *y***

In this case, we consider $m = 6$ and $n = 3 + 4y$ for a nonnegative integer $y$. We construct a Hamiltonian WDBC $W_{6,3}$ of $T_{6,3}$ and a Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$ as Fig. 2. Then, Fig. 3 shows how to apply these two cycles to form a Hamiltonian WDB of $T_{6,n}$ for $n = 3 + 4y$ and $y$ is a nonnegative integer: stack $H_{6,4}$ $y$ times on the top of $W_{6,3}$. Then, delete edge set $E_1 = \{(0, 3 + 4i)(1, 3 + 4i), (0, 6 + 4i)(1, 6 + 4i) \mid 0 \leq i \leq y - 1\} \cup \{(0, 0)(0, 2), (1, 0)(1, 2)\}$, and add edge set $E_2 = \{(0, 2 + 4i)(0, 3 + 4i) \cup (1, 2 + 4i)(1, 4 + 4i) \mid 0 \leq i \leq y - 1\} \cup \{(0, 0)(0, n - 1), (1, 0)(1, n - 1)\}$.

After these steps, a Hamiltonian cycle $W_{6,n}$ on $T_{6,n}$ is constructed, where $|E_1(W_{6,n})| = 10 + (14 - 2)y = 10 + 12y$ and $|E_2(W_{6,n})| = 8 + (10 + 2)y = 8 + 12y$. Consequently, $||E_1(W_{6,n})| - |E_2(W_{6,n})|| = 2$, $W_{6,n}$ satisfies the definition of Hamiltonian WDBC.

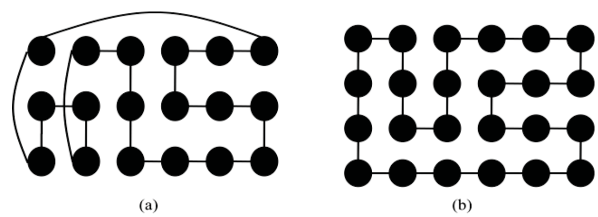


Fig. 2: (a) A Hamiltonian WDBC $W_{6,3}$ on $T_{6,3}$. (b) A Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$

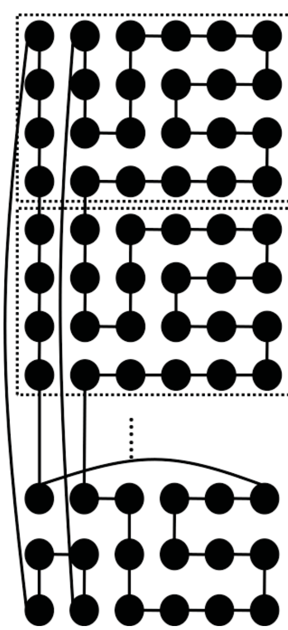

Fig. 3: A Hamiltonian WDBC $W_{6,n}$ on $T_{6,n}$

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For $m > 6$, we want to construct a Hamiltonian DBC for $T_{4,n}$. At first, we give a Hamiltonian DBC $B_{4,3}$ of $T_{4,3}$ and a Hamiltonian cycle $H_{4,4}$ on $T_{4,4}$ as Fig. 4. Besides, Fig. 5 shows how to use these two cycles to form a Hamiltonian WDB of $T_{4,n}$ for $n = 3 + 4y$ and $y$ is a nonnegative integer: stack $H_{4,4}$ $y$ times on the top of $B_{4,3}$. Then, delete edge set $E_3 = \{(2, 3 + 4i)(3, 3 + 4i), (2, 6 + 4i)(3, 6 + 4i) \mid 0 \le i \le y - 1\} \cup \{(2, 0)(2, 2), (3, 0)(3, 2)\}$, and add edge set $E_4 = \{(2, 2 + 4i)(2, 3 + 4i), (3, 2 + 4i)(3, 3 + 4i) \mid 0 \le i \le y - 1\} \cup \{(2, 0)(2, n - 1), (3, 0)(3, n - 1)\}$. After these steps, a Hamiltonian cycle $B_{4,n}$ of $T_{4,n}$ is generated, whose $|E_1(B_{4,n})| = 6 + (10 - 2)y = 6 + 8y$ and $|E_2(B_{4,n})| = 6 + (6 + 2)y = 6 + 8y$. Consequently, $||E_1(B_{4,n})| - |E_2(B_{4,n})|| = 0$, $B_{4,n}$ satisfies the definition of the Hamiltonian DBC.

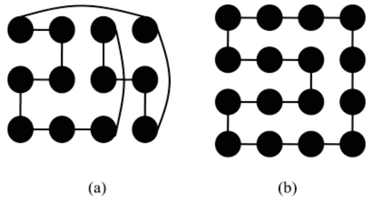



Fig. 4: (a) A Hamiltonian DBC $B_{4,3}$ of $T_{4,3}$. (b) A Hamiltonian cycle $H_{4,4}$ of $T_{4,4}$

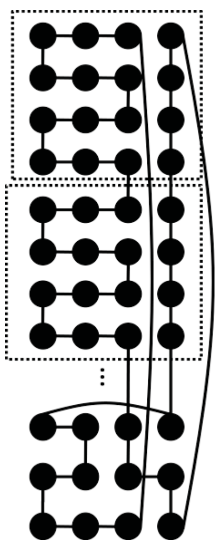


Fig. 5: A Hamiltonian DBC $B_{4,n}$ of $T_{4,n}$

Finally, let $x = (m - 6) / 4$, put $B_{4,n}$ for $x$ times on right side of $W_{6,n}$. Then, delete edge set $E_5 = \{(6 + 4i, 2)(9 + 4i, 2) \mid 0 \le i \le x - 1\} \cup \{(0, 2)(5, 2)\}$, and add edge set $E_6 = \{(5 + 4i, 2)(6 + 4i, 2) \mid 0 \le i \le x - 1\} \cup \{(0, 2)(m - 1, 2)\}$, show in Fig. 6. After these steps, a Hamiltonian cycle $W_{m,n}$ on $T_{m,n}$ for $m$ mod 4 = 2 and $n = 3 + 4y$ for a nonnegative integer $y$ is constructed, where $|E_1(W_{m,n})| = (10 + 12y) + (6 + 8y)x = 8xy + 6x + 12y + 10$ and $|E_2(W_{m,n})| = (8 + 12y) + (6 + 8y)x = 8xy + 6x + 12y + 8$. Consequently, $||E_1(W_{m,n})| - |E_2(W_{m,n})|| = 2$, $W_{m,n}$ satisfies the definition of weakly dimension-balanced Hamiltonian cycle.

**Case 2**. ***m* mod 4 = 2, *n* = 5 + 4*y* for nonnegative integer *y***

In this case, at first, we consider $m = 6$ and $n = 5 + 4y$ for a nonnegative integer $y$. We construct a WDBC $W_{6,5}$ on $T_{6,5}$ and a Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$ as shown in Fig. 7. Besides, Fig. 8 shows how to apply these two cycles to construct a Hamiltonian WDB of $T_{6,n}$ for $n = 5 + 4y$ and $y$ is a nonnegative integer: stack $H_{6,4}$ $y$ times on the top of $W_{6,5}$. Then, delete edge set $E_7 = \{(0, 5 + 4i)(1, 5 + 4i), (0, 8 + 4i)(1, 8 + 4i) \mid 0 \le i \le y - 1\} \cup \{(0, 0)(0, 4), (1, 0)(1, 4)\}$, and add edge set $E_8 = \{(0, 4 + 4i)(0, 5 + 4i), (1, 4 + 4i)(1, 5 + 4i) \mid 0 \le i \le y - 1\} \cup \{(0, 0)(0, n - 1), (1, 0)(1, n - 1)\}$. After these steps, a Hamiltonian cycle $W_{6,n}$ on $T_{6,n}$ is constructed, where $|E_1(W_{6,n})| = 16 + (14 - 2)y = 16 + 12y$ and $|E_2(W_{6,n})| = 14 + (10 + 2)y = 14 + 12y$. Consequently, $||E_1(W_{6,n})| - |E_2(W_{6,n})|| = 2$, $W_{6,n}$ satisfies the definition of weakly dimension-balanced Hamiltonian cycle.

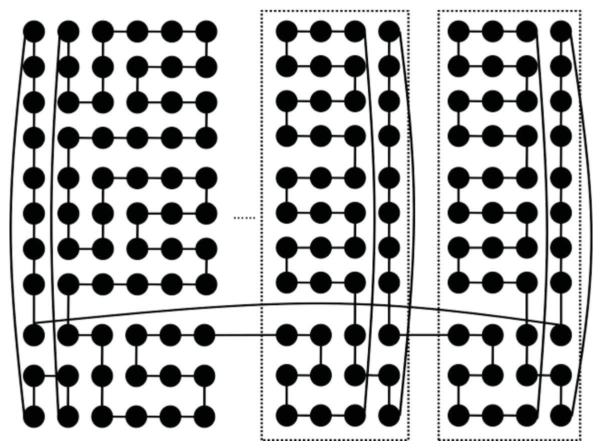


Fig. 6: A Hamiltonian WDBC on $T_{m,n}$ for $m$ mod 4 = 2 and $n = 3 + 4y$ for nonnegative integer $y$

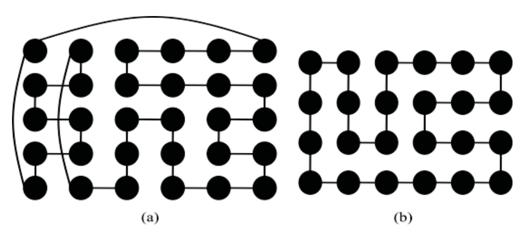



Fig. 7: (a) A Hamiltonian WDBC $W_{6,5}$ on $T_{6,5}$. (b) A Hamiltonian cycle $H_{6,4}$ on $T_{6,4}$

When $m > 6$, we construct a Hamiltonian DBC for $T_{4,n}$. At first, we give a Hamiltonian DBC $B_{4,5}$ of $T_{4,5}$ and a

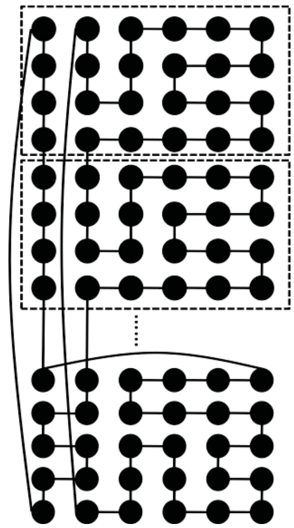
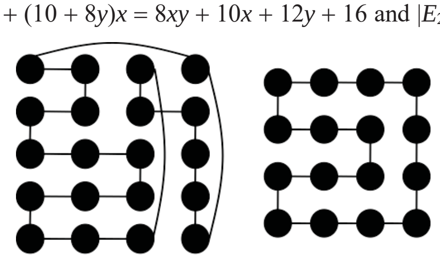


Fig. 8: A Hamiltonian WDBC $W_{6,n}$ on $T_{6,n}$

Hamiltonian cycle $H_{4,4}$ on $T_{4,4}$ as Fig. 9. Besides, Fig. 10 shows how to use these two cycles to form a Hamiltonian WDB of $T_{4,n}$ for $n = 5 + 4y$ and $y$ is a nonnegative integer: stack $H_{4,4}$ $y$ times on the top of $B_{4,5}$. Then, delete edge set $E_9 = \{(2, 5 + 4i)(3, 5 + 4i), (2, 8 + 4i)(3, 8 + 4i) \mid 0 \le i \le y - 1\} \cup \{(2, 0)(2, 4), (3, 0)(3, 4)\}$, and add edge set $E_{10} = \{(2, 4 + 4i)(2, 5 + 4i), (3, 4 + 4i)(3, 5 + 4i) \mid 0 \le i \le y - 1\} \cup \{(2, 0)(2, n - 1), (3, 0)(3, n - 1)\}$. After these steps, a Hamiltonian cycle $B_{4,n}$ of $T_{4,n}$ is generated, whose $|E_1(B_{4,n})| = 10 + (10 - 2)y = 10 + 8y$ and $|E_2(B_{4,n})| = 10 + (6 + 2)y = 10 + 8y$. Consequently, $||E_1(B_{4,n})| - |E_2(B_{4,n})|| = 0$, $B_{4,n}$ satisfies the definition of the Hamiltonian DBC.

Finally, let $x = (m - 6) / 4$, put $B_{4,n}$ for $x$ times on right side of $W_{6,n}$. Then, delete edge set $E_{11} = \{(6 + 4i, 4)(9 + 4i, 4) \mid 0 \le i \le x - 1\} \cup \{(0, 4)(5, 4)\}$, and add edge set $E_{12} = \{(5 + 4i, 4)(6 + 4i, 4) \mid 0 \le i \le x - 1\} \cup \{(0, 4)(m - 1, 4)\}$, as shown in Fig. 11. After these steps, a Hamiltonian cycle $W_{m,n}$ on $T_{m,n}$ for $m$ mod 4 = 2 and $n = 5 + 4y$ for a nonnegative integer $y$ is generated, whose $|E_1(W_{m,n})| = (16 + 12y) + (10 + 8y)x = 8xy + 10x + 12y + 16$ and $|E_2(W_{m,n})| = (14 + 12y) + (10 + 8y)x = 8xy + 10x + 12y + 14$. Consequently, $||E_1(W_{m,n})| - |E_2(W_{m,n})|| = 2$, $W_{m,n}$ satisfies the definition of weakly dimension-balanced Hamiltonian cycle. □



(a) (b)

Fig. 9: (a) A Hamiltonian DBC $B_{4,5}$ of $T_{4,5}$. (b) A Hamiltonian cycle $H_{4,4}$ of $T_{4,4}$.

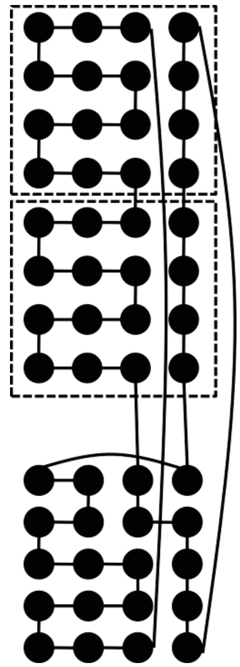


Fig. 10: A Hamiltonian DBC $B_{4,n}$ of $T_{4,n}$

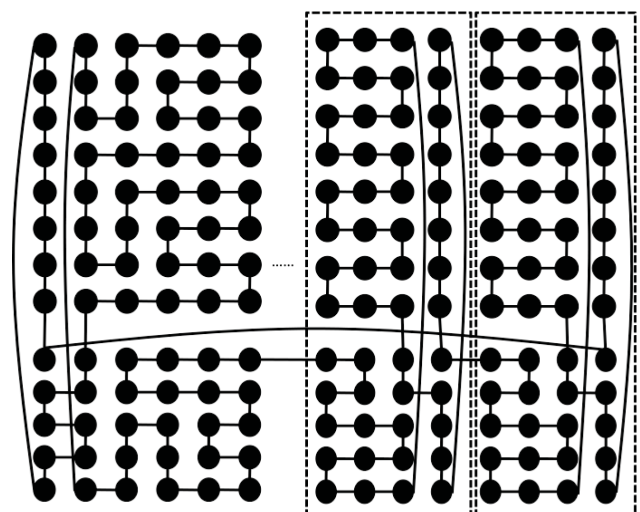


Fig. 11: A Hamiltonian WDBC on $T_{m,n}$ for $m$ mod 4 = 2 and $n = 5 + 4y$ for a nonnegative integer $y$.

By Theorem 6 and Corollary 1, Corollary 2 is the main result of this paper.

**Corollary 2.** *For any $m, n \ge 3$, toroidal mesh graph $T_{m,n}$ is weakly DB Hamiltonian.*

## IV. Conclusions

In this paper, we proved that the toroidal mesh graph $T_{m,n}$ contains a weakly dimension-balanced Hamiltonian cycle, where $m$, $n \geq 3$ and $mn$ mod 4 = 2. Referring to previous research works, we have concluded that $T_{m,n}$ contains a Hamiltonian WDBC for any $m$, $n \geq 3$. Thus, $T_{m,n}$ is weakly DB Hamiltonian.

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