# A Fully Adaptive Minimal Routing Algorithm in a Crossed Cube

**Kaito Satoh**[1], **Kousuke Mouri**[2], **and Keiichi Kaneko**[2]
[1]Graduate School of Engineering, Tokyo University of Agriculture and Technology, Koganei-shi, Tokyo, Japan
[2]Institute of Engineering, Tokyo University of Agriculture and Technology, Koganei-shi, Tokyo, Japan

**Abstract**—*A crossed cube can connect the same number of vertices as a hypercube while keeping the diameter almost half of the hypercube. In this paper, we propose an algorithm that can classify the neighboring vertices of an arbitrary vertex into those on the shortest paths, the sidetrack paths, and the backtrack paths to the destination vertex. The algorithm takes time that is proportional to the dimension of the crossed cube. By using the algorithm, we can achieve the fully adaptive minimal routing in a crossed cube.*

**Keywords:** Adaptive Routing, Hypercube, Interconnection Networks, Massively Parallel Systems, Parallel Processing

## 1. Introduction

Instead of sequential computation, parallel processing systems, especially massively parallel systems are focused on recently. Along this paradigm sift, many people are paying attention to the topologies for interconnection networks of massively parallel systems that have been proposed so far. The topologies are mainly divided into two categories. That is, the first one is based on the vertices of permutations such as the star graph [1], [2], the rotator graph [3], [4], the pancake graph [5], [6], the substring reversal graph [7], the bubble-sort graph [8], [9], and so on. The second one is based on the vertices of binary sequences such as the crossed cube [10], the möbius cube [11], [12], [13], the twisted cube [14], the locally twisted cube [15], the spined cube [16], [17], the dual-cube [18], [19], [20], and so on.

The crossed cube proposed by Efe [10] is one such topology. It is a variant of the hypercube [21], which was a very popular topology. A crossed cube can connect the same number of vertices as a hypercube while keeping the diameter almost half of the hypercube. Hence, it is enthusiastically studied [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36].

Efe has introduced a shortest path routing algorithm in an $n$-dimensional crossed cube $CQ_n$ [10]. For a current vertex and a destination vertex, the algorithm can find in $O(n)$ time at most two neighboring vertices of the current vertex that are on the shortest paths to the destination vertex. On the other hand, Chang et al. [22] have proposed another algorithm, which can generate a shortest path in $O(n)$ time in $CQ_n$. However, these algorithms are impractical because they cannot provide a sufficient number of selections in routing. Therefore, in this paper, we propose an algorithm in $CQ_n$. For a current vertex and a destination vertex in $CQ_n$, the algorithm classifies all neighboring vertices of the current vertex into those on the shortest paths, the sidetrack paths, and the backtrack paths to the destination vertex in $O(n)$ time.

The rest of this paper is structured as follows. In Section 2, we introduce necessary definitions and a lemma. In Section 3, we show a core lemma in this paper followed by the description of our algorithm based on it. In Section 4, we conclude our study and give future works.

## 2. Preliminaries

In this section, we introduce definitions of the crossed cubes and other requisite notions as well as an important lemma.

*Definition 1:* For a pair of 2 bits, $(u_1, u_0)$ and $(v_1, v_0)$, a relation $(u_1, u_0) \sim (v_1, v_0)$ holds if and only if either $(u_1, u_0) = (v_1, v_0) = (0, 0)$, $(u_1, u_0) = (v_1, v_0) = (1, 0)$, or $\{(u_1, u_0), (v_1, v_0)\} = \{(0, 1), (1, 1)\}$ holds.

*Definition 2:* An $n$-dimensional crossed cube $CQ_n$ consists of a vertex set $V(CQ_n) = \{0, 1\}^n$. For two vertices $\boldsymbol{u} = (u_{n-1}, u_{n-2}, \ldots, u_0), \boldsymbol{v} = (v_{n-1}, v_{n-2}, \ldots, v_0) \in V(CQ_n)$, the edge $(\boldsymbol{u}, \boldsymbol{v})$ is included in the edge set $E(CQ_n)$ if and only if $\exists k(\in \{0, 1, \ldots, n-1\})$ such that $u_i = v_i$ $(k+1 \leq \forall i \leq n-1)$, $u_k = \overline{v}_k$, $u_{k-1} = v_{k-1}$ (if $k$ is odd), and $(u_{2i+1}, u_{2i}) \sim (v_{2i+1}, v_{2i})$ $(0 \leq \forall i \leq \bar{k} - 1$ where $\bar{k} = \lfloor k/2 \rfloor)$. By using $k$, the neighboring vertex $\boldsymbol{v}$ is specified by $\boldsymbol{u}^{(k)}$.

Let $\boldsymbol{u} = (1, 0, 0, 1, 1, 0)$ be a vertex in $CQ_6$. Then, it has six neighboring vertices: $\boldsymbol{u}^{(5)} = (0, 0, 1, 1, 1, 0)$, $\boldsymbol{u}^{(4)} = (1, 1, 1, 1, 1, 0)$, $\boldsymbol{u}^{(3)} = (1, 0, 1, 1, 1, 0)$, $\boldsymbol{u}^{(2)} = (1, 0, 0, 0, 1, 0)$, $\boldsymbol{u}^{(1)} = (1, 0, 0, 1, 0, 0)$, and $\boldsymbol{u}^{(0)} = (1, 0, 0, 1, 1, 1)$. Figure 1 shows an example of a 4-dimensional crossed cube $CQ_4$ where a vertex $(u_3, u_2, u_1, u_0)$ is denoted by the juxtaposition of the elements $u_3u_2u_1u_0$ to save space.

Table 1 shows the comparison of an $n$-dimensional crossed cube $CQ_n$ with an $n$-dimensional hypercube $Q_n$, an $n$-dimensional 0-möbius cube 0-$MQ_n$, an $n$-dimensional 1-möbius cube 1-$MQ_n$, an $n$-dimensional twisted cube $TQ_n$, an $n$-dimensional locally twisted cube $LTQ_n$, an $n$-dimensional spined cube $SQ_n$, and a $(2n-1)$-dimensional dual-cube $DQ_n$. In the table, '#Verts.', 'Deg.', 'Reg.', and 'Sym.' represent 'Number of Vertices', 'Degree', 'Regularity', and 'Symmetry', respectively. 'Density' is defined by '#Verts.'/('Deg.' × 'Diameter'). Except for $DQ_n$, $CQ_n$ has

ISBN: 1-60132-487-1, CSREA Press ©

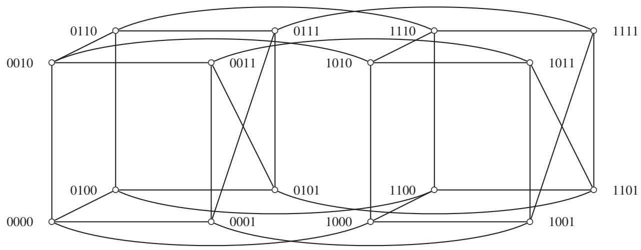


Fig. 1: An example of a 4-dimensional crossed cube $CQ_4$.

almost same density as other topologies, which is almost twice of $Q_n$. In addition, $CQ_n$ has the property of vertex and edge symmetry. On the other hand, $DQ_n$ attains the best density, which is much larger than the other topologies. However, $DQ_n$ has just vertex-symmetric property. Moreover, it lacks the property of incremental expandability, which is pointed out to be important as a topology for interconnection networks by Duato et al. [37].

Table 1: Comparison of $CQ_n$ with $Q_n$, 0-$MQ_n$, 1-$MQ_n$, $TQ_n$, $LTQ_n$, $SQ_n$, and $DQ_n$.

| Topology | #Verts. | Deg. | Diameter | Density | Reg. | Sym. |
|---|---|---|---|---|---|---|
| $CQ_n$ | $2^n$ | $n$ | $\lceil(n+1)/2\rceil$ | $2^n/n\lceil(n+1)/2\rceil$ | ✓ | ✓ |
| $Q_n$ | $2^n$ | $n$ | $n$ | $2^n/n^2$ | ✓ | ✓ |
| 0-$MQ_n$ | $2^n$ | $n$ | $\lceil(n+2)/2\rceil$ | $2^n/n\lceil(n+2)/2\rceil$ | ✓ | |
| 1-$MQ_n$ | $2^n$ | $n$ | $\lceil(n+1)/2\rceil$ | $2^n/n\lceil(n+1)/2\rceil$ | ✓ | |
| $TQ_n$ | $2^n$ | $n$ | $\lceil(n+1)/2\rceil$ | $2^n/n\lceil(n+1)/2\rceil$ | ✓ | |
| $LTQ_n$ | $2^n$ | $n$ | $\lceil(n+3)/2\rceil$ | $2^n/n\lceil(n+3)/2\rceil$ | ✓ | |
| $SQ_n$ | $2^n$ | $n$ | $\lceil n/3\rceil+3$ | $2^n/n(\lceil n/3\rceil+3)$ | ✓ | |
| $DQ_n$ | $2^{2n-1}$ | $n$ | $2n$ | $2^{2n-2}/n^2$ | ✓ | |

*Definition 3:* For two vertices $\boldsymbol{u}=(u_{n-1},u_{n-2},\ldots,u_0)$ and $\boldsymbol{v}=(v_{n-1},v_{n-2},\ldots,v_0)\in V(CQ_n)$, let $h$ $(0\le h\le n-1)$ be the bit position such that $u_i=v_i$ $(h+1\le\forall i\le n-1)$ and $u_h=\overline{v}_h$. Also, let $\hbar=\lfloor h/2\rfloor$ and $\tilde{n}=\lfloor(n-1)/2\rfloor$. Then, the function $\rho(\boldsymbol{u},\boldsymbol{v},j)$ $(0\le j\le\tilde{n})$ is defined as follows:

$$\rho(\boldsymbol{u},\boldsymbol{v},j)=\begin{cases}0 & (j>\hbar)\\ 2 & (j=\hbar,\ (u_{2\hbar+1},u_{2\hbar})=(\overline{v}_{2\hbar+1},\overline{v}_{2\hbar}))\\ 1 & (j=\hbar,\ (u_{2\hbar+1},u_{2\hbar})\neq(\overline{v}_{2\hbar+1},\overline{v}_{2\hbar}))\\ 0 & (j<\hbar,\ (u_{2j+1},u_{2j})\approx(v_{2j+1},v_{2j}))\\ 1 & (j<\hbar,\ (u_{2j+1},u_{2j})\not\approx(v_{2j+1},v_{2j}))\end{cases}$$

where for a pair of 2 bits, $(u_{2j+1},u_{2j})$ and $(v_{2j+1},v_{2j})$, a relation $(u_{2j+1},u_{2j})\approx(v_{2j+1},v_{2j})$ holds if and only if either one of the three conditions below holds:

(1) $(u_{2j+1},u_{2j})=(v_{2j+1},v_{2j})=(0,1)$ or $(1,1)$, and $\sum_{i=j+1}^{\tilde{n}}\rho(\boldsymbol{u},\boldsymbol{v},i)$ is even.

(2) $\{(u_{2j+1},u_{2j}),(v_{2j+1},v_{2j})\}=\{(0,1),(1,1)\}$, and $\sum_{i=j+1}^{\tilde{n}}\rho(\boldsymbol{u},\boldsymbol{v},i)$ is odd.

(3) $(u_{2j+1},u_{2j})=(v_{2j+1},v_{2j})=(0,0)$ or $(1,0)$.

By calculating from $j=\tilde{n}$ to $j=0$, $\rho(\boldsymbol{u},\boldsymbol{v},j)$ $(0\le j\le\tilde{n})$ can be obtained in $O(n)$ time. Figure 2 shows the algorithm. For example, for two vertices $\boldsymbol{u}=(0,1,1,1,0,1,0,1,1,1,1,0,1,1,1,0,0,1,1,1)$ and $\boldsymbol{v}=(0,1,1,1,1,0,0,1,1,0,0,1,1,1,1,0,1,1,0,1)$ in $CQ_{20}$, $h=15$ and $\hbar=7$. Then, from Definition 3, we can have $\rho(\boldsymbol{u},\boldsymbol{v},9)=\rho(\boldsymbol{u},\boldsymbol{v},8)=0$, $\rho(\boldsymbol{u},\boldsymbol{v},7)=2$, $\rho(\boldsymbol{u},\boldsymbol{v},6)=0$, $\rho(\boldsymbol{u},\boldsymbol{v},5)=\rho(\boldsymbol{u},\boldsymbol{v},4)=1$, $\rho(\boldsymbol{u},\boldsymbol{v},3)=\rho(\boldsymbol{u},\boldsymbol{v},2)=0$, $\rho(\boldsymbol{u},\boldsymbol{v},1)=1$, and $\rho(\boldsymbol{u},\boldsymbol{v},0)=0$.

```
procedure rho(u, v)
begin
  for j := 0 to ñ do
    r[j] := 0;
  s := 0;
  for j := ñ to 0 step -1 do begin
    if s = 0 then begin
      if (u_{2j+1}, u_{2j}) = (v̄_{2j+1}, v̄_{2j}) then
        r[j] := 2
      else if (u_{2j+1}, u_{2j}) <> (v_{2j+1}, v_{2j}) then
        r[j] := 1 end
    else if (u_{2j+1}, u_{2j}) ≉ (v_{2j+1}, v_{2j}) then
      r[j] := 1;
    s := s + r[j]
  end;
  return r
end
```

Fig. 2: Algorithm for calculation of $r[j]=\rho(\boldsymbol{u},\boldsymbol{v},j)$ $(\tilde{n}\ge j\ge 0)$.

*Lemma 1:* For two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $CQ_n$, let $\rho(\boldsymbol{u},\boldsymbol{v})=\sum_{j=0}^{\tilde{n}}\rho(\boldsymbol{u},\boldsymbol{v},j)$. Then, $d(\boldsymbol{u},\boldsymbol{v})=\rho(\boldsymbol{u},\boldsymbol{v})$ holds.
(Proof) See the proof by Chang et al. [22].

From Lemma 1, we can calculate the distance between an arbitrary pair of vertices in $O(n)$ time. The distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ in the above example is equal to $\sum_{j=0}^{9}\rho(\boldsymbol{u},\boldsymbol{v},j)=5$.

## 3. Classification of Neighboring Vertices

In this section, for an arbitrary vertex $\boldsymbol{u}$ and a destination vertex $\boldsymbol{v}$ in $CQ_n$, we give a method to classify the elements of the neighboring vertex set $N(\boldsymbol{u})$ of $\boldsymbol{u}$ into three subsets in $O(n)$ time depending on the distances to $\boldsymbol{v}$: $N_{-1}(\boldsymbol{u},\boldsymbol{v})=\{\boldsymbol{n}\mid\boldsymbol{n}\in N(\boldsymbol{u}),d(\boldsymbol{n},\boldsymbol{v})=d(\boldsymbol{u},\boldsymbol{v})-1\}$, $N_0(\boldsymbol{u},\boldsymbol{v})=\{\boldsymbol{n}\mid\boldsymbol{n}\in N(\boldsymbol{u}),d(\boldsymbol{n},\boldsymbol{v})=d(\boldsymbol{u},\boldsymbol{v})\}$, and $N_{+1}(\boldsymbol{u},\boldsymbol{v})=\{\boldsymbol{n}\mid\boldsymbol{n}\in N(\boldsymbol{u}),d(\boldsymbol{n},\boldsymbol{v})=d(\boldsymbol{u},\boldsymbol{v})+1\}$.

*Lemma 2:* For two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $CQ_n$, let $\boldsymbol{u}^{(k)}$ be a neighboring vertex of $\boldsymbol{u}$. Then, $\rho(\boldsymbol{u}^{(k)},\boldsymbol{v},j)=\rho(\boldsymbol{u},\boldsymbol{v},j)$ holds for $j$ $(\tilde{n}\ge j\ge\bar{k}+1)$ where $\bar{k}=\lfloor k/2\rfloor$.
(Proof) Let $\boldsymbol{u}=(u_{n-1},u_{n-2},\ldots,u_0)$ and $\boldsymbol{u}^{(k)}=(u'_{n-1},u'_{n-2},\ldots,u'_0)$. Then, $(u'_{2j+1},u'_{2j})=(u_{2j+1},u_{2j})$ holds for $j$ $(\tilde{n}\ge j\ge\bar{k}+1)$. Hence, $\rho(\boldsymbol{u}^{(k)},\boldsymbol{v},j)=\rho(\boldsymbol{u},\boldsymbol{v},j)$ holds. See Figure 3. □

*Lemma 3:* For two vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $CQ_n$, let $\boldsymbol{u}^{(k)}$ $(0\le k\le n-1)$ be the neighboring vertices of $\boldsymbol{u}$. If $\rho(\boldsymbol{u},\boldsymbol{v},j)$ $(0\le j\le\tilde{n})$ are all given, we can calculate $\rho(\boldsymbol{u}^{(k)},\boldsymbol{v})$ $(0\le k\le n-1)$ in $O(n)$ time.
(Proof) For $\boldsymbol{u}=(u_{n-1},u_{n-2},\ldots,u_0)$ and $\boldsymbol{v}=(v_{n-1},v_{n-2},\ldots,v_0)$, let $h=\max\{i\mid u_i=\overline{v}_i\}$, $\hbar=\lfloor h/2\rfloor$,

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ |
|---|---|---|---|
| $\boldsymbol{u}$ | $(u_{n-1}, u_{n-2})$ | $\cdots$ | $(u_{2\bar{k}+3}, u_{2\bar{k}+2})$ |
| $\boldsymbol{u}^{(k)}$ | $(u_{n-1}, u_{n-2})$ | $\cdots$ | $(u_{2\bar{k}+3}, u_{2\bar{k}+2})$ |
| $\Delta\rho$ | 0 | $\cdots$ | 0 |

Fig. 3: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ for $j$ ($\tilde{n} \geq j \geq \bar{k}+1$).

$\bar{k} = \lfloor k/2 \rfloor$, and $\boldsymbol{u}^{(k)} = (u'_{n-1}, u'_{n-2}, \ldots, u'_0)$. Then, we prove this lemma by three cases: $\bar{k} < \hbar$, $\bar{k} = \hbar$, and $\bar{k} > \hbar$.

**Case 1** ($\bar{k} < \hbar$) From Definition 3, if $\bar{k} < \hbar$, $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 0$ or 1.

**Case 1a** First, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 0$. From Definition 3, if $\bar{k} < \hbar$ and $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 0$, $(u_{2\bar{k}+1}, u_{2\bar{k}}) \approx (v_{2\bar{k}+1}, v_{2\bar{k}})$ holds. Also, from Lemma 2, $\sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) = \sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$ holds. Since $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (u_{2\bar{k}+1}, \bar{u}_{2\bar{k}})$ or $(\bar{u}_{2\bar{k}+1}, u_{2\bar{k}})$ (see Table 2), $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \not\approx (v_{2\bar{k}+1}, v_{2\bar{k}})$ holds from Definition 3. Therefore, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 1$ holds, and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) + 1$ holds.

Table 2: $(u'_{2\bar{k}+1}, u'_{2\bar{k}})$ in $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k})$.

| $(u_{2\bar{k}+1}, u_{2\bar{k}})$ | $(v_{2\bar{k}+1}, v_{2\bar{k}})$ | $\sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ | $(u'_{2\bar{k}+1}, u'_{2\bar{k}})$ | $\sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$ |
|---|---|---|---|---|
| (0,0) | (0,0) | even or odd | (0,1) or (1,0) | even or odd |
| (0,1) | (0,1) | even | (0,0) or (1,1) | even |
| (0,1) | (1,1) | odd | (0,0) or (1,1) | odd |
| (1,0) | (1,0) | even or odd | (0,0) or (1,1) | even or odd |
| (1,1) | (0,1) | odd | (0,1) or (1,0) | odd |
| (1,1) | (1,1) | even | (0,1) or (1,0) | even |

Now, let us consider about $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}-1)$. From Definition 2, Table 3 shows the relationship $(u_{2j+1}, u_{2j}) \sim (u'_{2j+1}, u'_{2j})$ ($\bar{k}-1 \geq j \geq 0$). From Lemma 2 and the fact that $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) + 1$ just proved above, $\sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) + 1$ holds. Hence, from Definition 3, $(u'_{2\bar{k}-1}, u'_{2\bar{k}-2}) \approx (v_{2\bar{k}-1}, v_{2\bar{k}-2})$ holds if and only if $(u_{2\bar{k}-1}, u_{2\bar{k}-2}) \approx (v_{2\bar{k}-1}, v_{2\bar{k}-2})$ holds. Therefore, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}-1) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}-1)$ holds. Similarly, for all $j$ ($\bar{k}-1 \geq j \geq 0$), $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$. From above discussion, $\sum_{j=0}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=0}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) + 1$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \rho(\boldsymbol{u}, \boldsymbol{v}) + 1$ holds. See Figure 4.

Table 3: Relationship $(u_{2j+1}, u_{2j}) \sim (u'_{2j+1}, u'_{2j})$ ($\bar{k}-1 \geq j \geq 0$).

| $(u_{2j+1}, u_{2j})$ | $(u'_{2j+1}, u'_{2j})$ |
|---|---|
| (0,0) | (0,0) |
| (0,1) | (1,1) |
| (1,0) | (1,0) |
| (1,1) | (0,1) |

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

Fig. 4: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 1a.

**Case 1b** Next, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$ and $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \approx (v_{2\bar{k}+1}, v_{2\bar{k}})$. From Definition 3, if $\bar{k} < \hbar$ and $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \approx (v_{2\bar{k}+1}, v_{2\bar{k}})$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 0$ holds. That is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) - 1$ holds. In addition, from Lemma 2, $\sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ holds. Furthermore, similarly to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k}-1 \geq j \geq 0$) can be proved. From above discussion, $\sum_{j=0}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=0}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) - 1$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \rho(\boldsymbol{u}, \boldsymbol{v}) - 1$ holds. See Figure 5.

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

Fig. 5: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 1b.

**Case 1c** Finally, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$ and $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \not\approx (v_{2\bar{k}+1}, v_{2\bar{k}})$. From Definition 3, if $\bar{k} < \hbar$ and $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \not\approx (v_{2\bar{k}+1}, v_{2\bar{k}})$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 1$ holds. That is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k})$ holds. Now, let $S$ be $\{j \mid \bar{k} > j \text{ and } u_{2j} = v_{2j} = 1\}$. If $S = \emptyset$, the relationship among $(u_{2j+1}, u_{2j})$, $(v_{2j+1}, v_{2j})$, and $(u'_{2j+1}, u'_{2j})$ ($\bar{k}-1 \geq j \geq 0$) is shown in Table 4, and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ holds. If $S \neq \emptyset$, let $l = \max S$.

Table 4: Relationship among $(u_{2j+1}, u_{2j})$, $(v_{2j+1}, v_{2j})$, and $(u'_{2j+1}, u'_{2j})$ ($\bar{k}-1 \geq j \geq 0$) with $\rho(\boldsymbol{u}, \boldsymbol{v}, j)$ and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$.

| $(u_{2j+1}, u_{2j})$ | $(v_{2j+1}, v_{2j})$ | $(u'_{2j+1}, u'_{2j})$ | $\rho(\boldsymbol{u}, \boldsymbol{v}, j)$ | $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$ |
|---|---|---|---|---|
| (0,0) | (0,0) | (0,0) | 0 | 0 |
| (0,0) | (0,1) | (0,0) | 1 | 1 |
| (0,0) | (1,0) | (0,0) | 1 | 1 |
| (0,0) | (1,1) | (0,0) | 1 | 1 |
| (0,1) | (0,0) | (1,1) | 1 | 1 |
| (0,1) | (1,0) | (1,1) | 1 | 1 |
| (1,0) | (0,0) | (1,0) | 1 | 1 |
| (1,0) | (0,1) | (1,0) | 1 | 1 |
| (1,0) | (1,0) | (1,0) | 0 | 0 |
| (1,0) | (1,1) | (1,0) | 1 | 1 |
| (1,1) | (0,0) | (0,1) | 1 | 1 |
| (1,1) | (1,0) | (0,1) | 1 | 1 |

Then, similarly to the case with $S = \emptyset$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k}-1 \geq j \geq l+1$) can be proved. From Lemma 2, $\sum_{j=l+1}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=l+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ holds. If $(u_{2l+1}, u_{2l}) = (0, 1)$, $(u'_{2l+1}, u'_{2l}) = (1, 1)$ holds. Otherwise, that is, if $(u_{2l+1}, u_{2l}) = (1, 1)$, $(u'_{2l+1}, u'_{2l}) = (0, 1)$ holds. Hence, from Definition 3,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, l) = \begin{cases} 0 & (\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1) \\ 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0) \end{cases}$$

holds. Based on the similar discussion to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($l-1 \geq j \geq 0$) can be proved. From above discussion,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \begin{cases} \rho(\boldsymbol{u}, \boldsymbol{v}) & (S = \emptyset) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) + 1 & (S \neq \emptyset, \rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) - 1 & (S \neq \emptyset, \rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1) \end{cases}$$

$S = \emptyset$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $0$ | $\sum\Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | 0 |

$S \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | $0$ | $\sum\Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

$S \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | $0$ | $\sum\Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | 0 | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

Fig. 6: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 1c.

See Figure 6.

**Case 2** ($\bar{k} = \hbar$) From Definition 3, if $\bar{k} = \hbar$, $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$ or 2.

**Case 2a** First, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 2$. From Definition 3, if $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 2$, $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$. Because $u_k = \bar{u}'_k$, $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, v_{2\bar{k}})$ or $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (v_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$. Hence, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 1$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) - 1$ holds. Then, from Lemma 2, $\sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) - 1$. Moreover, similarly to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k}-1 \geq j \geq 0$) holds. From above discussion, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \rho(\boldsymbol{u}, \boldsymbol{v}) - 1$ holds. See Figure 7.

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $0$ | $\sum\Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

Fig. 7: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 2a.

**Case 2b** Next, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$ and $u_k = v_k$. From Definition 3, if $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$, $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, v_{2\bar{k}})$ or $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (v_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$ holds. In addition, $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$ because $u_k = v_k$. Hence, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 2$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) + 1$ holds. Then, from Lemma 2, $\sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) + 1$. Moreover, similarly to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k}-1 \geq j \geq 0$) holds. From above discussion, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \rho(\boldsymbol{u}, \boldsymbol{v}) + 1$ holds. See Figure 8.

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $0$ | $\sum!\Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

Fig. 8: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 2b.

**Case 2c** Finally, let us assume $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$ and $u_k = \bar{v}_k$. From Definition 3, if $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 1$, $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, v_{2\bar{k}})$ or $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (v_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$ holds. $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (v_{2\bar{k}+1}, v_{2\bar{k}})$ because $u'_k = \bar{u}_k = v_k$. Hence, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 0$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) - 1$ holds. Now, let $S$ be $\{j \mid \bar{k} > j \text{ and } \rho(\boldsymbol{u}, \boldsymbol{v}, j) = 1\}$. if $S = \emptyset$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k}-1 \geq j \geq 0$) from Definition 3. If $S \neq \emptyset$, let $l = \max S$. Then, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k} - 1 \geq j \geq l + 1$) holds. The relationship among $(u_{2j+1}, u_{2j})$, $(v_{2j+1}, v_{2j})$, and $(u'_{2j+1}, u'_{2j})$ ($\bar{k}-1 \geq j \geq 0$) is shown in Table 5. From

Table 5: Relationship among $(u_{2j+1}, u_{2j})$, $(v_{2j+1}, v_{2j})$, and $(u'_{2j+1}, u'_{2j})$ ($\bar{k} - 1 \geq j \geq 0$) with $\rho(\boldsymbol{u}, \boldsymbol{v}, j)$ and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$.

| $(u_{2j+1}, u_{2j})$ | $(v_{2j+1}, v_{2j})$ | $(u'_{2j+1}, u'_{2j})$ | $\rho(\boldsymbol{u}, \boldsymbol{v}, j)$ | $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j)$ |
|---|---|---|---|---|
| (0,0) | (0,1) | (0,0) | 1 | 1 |
| (0,0) | (1,0) | (0,0) | 1 | 1 |
| (0,0) | (1,1) | (0,0) | 1 | 2 |
| (0,1) | (0,0) | (1,1) | 1 | 2 |
| (0,1) | (1,0) | (1,1) | 1 | 1 |
| (0,1) | (1,1) | (1,1) | 1 | 1 |
| (1,0) | (0,0) | (1,0) | 1 | 1 |
| (1,0) | (0,1) | (1,0) | 1 | 2 |
| (1,0) | (1,1) | (1,0) | 1 | 1 |
| (1,1) | (0,0) | (0,1) | 1 | 1 |
| (1,1) | (0,1) | (0,1) | 1 | 1 |
| (1,1) | (1,0) | (0,1) | 1 | 2 |

Table 5, if $u_{2l+1} \neq v_{2l+1} \oplus v_{2l}$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, l) = \rho(\boldsymbol{u}, \boldsymbol{v}, l)$. From Lemma 2 and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) - 1$ as shown above, $\sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) - 1$. Moreover, similarly to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($l - 1 \geq j \geq 0$) holds. Otherwise, if $u_{2l+1} = v_{2l+1} \oplus v_{2l}$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, l) = \rho(\boldsymbol{u}, \boldsymbol{v}, l) + 1$. From Lemma 2 and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) - 1$ as shown above, $\sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\bar{k}}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$. Moreover, similarly to Case 1c, let $T$ be $\{j \mid j \leq \bar{k} - 1 \text{ and } u_{2j} = v_{2j} = 1\}$. If $T = \emptyset$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($0 \leq j \leq l - 1$) holds. Otherwise, if $T \neq \emptyset$, let $m = \max T$. Then,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, m) = \begin{cases} \rho(\boldsymbol{u}, \boldsymbol{v}, m) + 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, m) = 0) \\ \rho(\boldsymbol{u}, \boldsymbol{v}, m) - 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, m) = 1) \end{cases}$$

and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ ($\bar{k} - 1 \geq j \geq 0$) hold. From above discussion,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \begin{cases} \rho(\boldsymbol{u}, \boldsymbol{v}) - 1 & (S = \emptyset) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) - 1 & (S \neq \emptyset, u_{2l+1} \neq v_{2l+1} \oplus v_{2l}) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) & (S \neq \emptyset, T = \emptyset, u_{2l+1} = v_{2l+1} \oplus v_{2l}) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) - 1 & (S \neq \emptyset, T \neq \emptyset, u_{2l+1} = v_{2l+1} \oplus v_{2l}, \rho(\boldsymbol{u}, \boldsymbol{v}, m) = 1) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) + 1 & (S \neq \emptyset, T \neq \emptyset, u_{2l+1} = v_{2l+1} \oplus v_{2l}, \rho(\boldsymbol{u}, \boldsymbol{v}, m) = 0) \end{cases}$$

holds. See Figure 9.

**Case 3** ($\bar{k} > \hbar$) If $\bar{k} > \hbar$, from Definition 3, $\rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) = 0$. Because $u'_k = \bar{u}_k$ and $(u_{2\bar{k}+1}, u_{2\bar{k}}) = (v_{2\bar{k}+1}, v_{2\bar{k}})$, either $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (v_{2\bar{k}+1}, \bar{v}_{2\bar{k}})$ or $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) = (\bar{v}_{2\bar{k}+1}, v_{2\bar{k}})$ holds. Therefore, from Definition 3, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = 1$, that is, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \bar{k}) = \rho(\boldsymbol{u}, \boldsymbol{v}, \bar{k}) + 1$ holds. With $j = \bar{k} - 1$, $\sum_{i=j+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, i) = 1$ holds. Because $(u_{2j+1}, u_{2j}) = (v_{2j+1}, v_{2j}) \sim (u'_{2j+1}, u'_{2j})$, $(u'_{2j+1}, u'_{2j})$ and $(v_{2j+1}, v_{2j})$ satisfy either of (2) or (3) in Definition 3. Hence, $(u'_{2j+1}, u'_{2j}) \approx (v_{2j+1}, v_{2j})$ holds, and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = 0$. Similarly, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j) = 0$ holds for $j$ ($\bar{k}-1 \geq$

$S = \emptyset$ or $S \neq \emptyset$, $u_{2l+1} \neq v_{2l+1} \oplus v_{2l}$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

$S \neq \emptyset$, $u_{2l+1} = v_{2l+1} \oplus v_{2l}$, $T = \emptyset$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | 0 |

$S \neq \emptyset$, $u_{2l+1} = v_{2l+1} \oplus v_{2l}$, $T \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | $m+1$ | $m$ | $m-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

$S \neq \emptyset$, $u_{2l+1} = v_{2l+1} \oplus v_{2l}$, $T \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | $m+1$ | $m$ | $m-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

Fig. 9: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 2c.

$\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 1$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

$\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2$, $S = \emptyset$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $\hbar+1$ | $\hbar$ | $\hbar-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | 0 |

$\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2$, $S \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $\hbar+1$ | $\hbar$ | $\hbar-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $-1$ |

$\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2$, $S \neq \emptyset$, $\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0$

| $j$ | $\tilde{n}$ | $\cdots$ | $\bar{k}+1$ | $\bar{k}$ | $\bar{k}-1$ | $\cdots$ | $\hbar+1$ | $\hbar$ | $\hbar-1$ | $\cdots$ | $l+1$ | $l$ | $l-1$ | $\cdots$ | 0 | $\sum \Delta\rho$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta\rho$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $-1$ | 0 | $\cdots$ | 0 | $+1$ | 0 | $\cdots$ | 0 | $+1$ |

Fig. 10: $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) - \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ in Case 3c.

$j \geq \hbar+1$). From Lemma 2, $\sum_{j=\hbar+1}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = 1$ holds. Hence, the relationship among $(u_{2\hbar+1}, u_{2\hbar})$, $(v_{2\hbar+1}, v_{2\hbar})$, and $(u'_{2\hbar+1}, u'_{2\hbar})$ is shown in Table 6. From Table 6, if

Table 6: Relationship among $(u_{2\hbar+1}, u_{2\hbar})$, $(v_{2\hbar+1}, v_{2\hbar})$, and $(u'_{2\hbar+1}, u'_{2\hbar})$ with $\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar)$ and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \hbar)$.

| $(u_{2\hbar+1}, u_{2\hbar})$ | $(v_{2\hbar+1}, v_{2\hbar})$ | $(u'_{2\hbar+1}, u'_{2\hbar})$ | $\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar)$ | $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \hbar)$ |
|---|---|---|---|---|
| (0,0) | (0,1) | (0,0) | 1 | 1 |
| (0,0) | (1,0) | (0,0) | 1 | 1 |
| (0,0) | (1,1) | (0,0) | 2 | 1 |
| (0,1) | (0,0) | (1,1) | 1 | 1 |
| (0,1) | (1,0) | (1,1) | 2 | 1 |
| (0,1) | (1,1) | (1,1) | 1 | 1 |
| (1,0) | (0,0) | (1,0) | 1 | 1 |
| (1,0) | (0,1) | (1,0) | 2 | 1 |
| (1,0) | (1,1) | (1,0) | 1 | 1 |
| (1,1) | (0,0) | (0,1) | 2 | 1 |
| (1,1) | (0,1) | (0,1) | 1 | 1 |
| (1,1) | (1,0) | (0,1) | 1 | 1 |

$\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 1$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \hbar) = \rho(\boldsymbol{u}, \boldsymbol{v}, \hbar)$ holds. Hence, from Lemma 2, $\sum_{j=\hbar}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\hbar}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j) + 1$ holds. In addition, similarly to Case 1a, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ $(\hbar - 1 \geq j \geq 0)$ holds. From Table 6, if $\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, \hbar) = \rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) - 1$ holds. Hence, from Lemma 2, $\sum_{j=\hbar}^{\tilde{n}} \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \sum_{j=\hbar}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ holds. In addition, similarly to Case 1c, let $S = \{j \mid j \leq \hbar - 1, u_{2j} = v_{2j} = 1\}$. If $S = \emptyset$, $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, j) = \rho(\boldsymbol{u}, \boldsymbol{v}, j)$ $(l - 1 \geq j \geq 0)$ holds. Otherwise, if $S \neq \emptyset$, let $l = \max S$. Then,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, l) = \begin{cases} \rho(\boldsymbol{u}, \boldsymbol{v}, l) + 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0) \\ \rho(\boldsymbol{u}, \boldsymbol{v}, l) - 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1) \end{cases}$$

and $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}, i) = \rho(\boldsymbol{u}, \boldsymbol{v}, i)$ $(\bar{k} - 1 \geq i \neq l \geq 0)$ hold. From above discussion,

$$\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) = \begin{cases} \rho(\boldsymbol{u}, \boldsymbol{v}) + 1 & (\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 1) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) & (\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2, S = \emptyset) \\ \rho(\boldsymbol{u}, \boldsymbol{v}) + 1 & \\ (\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2, S \neq \emptyset, \rho(\boldsymbol{u}, \boldsymbol{v}, l) = 0) & \\ \rho(\boldsymbol{u}, \boldsymbol{v}) - 1 & \\ (\rho(\boldsymbol{u}, \boldsymbol{v}, \hbar) = 2, S \neq \emptyset, \rho(\boldsymbol{u}, \boldsymbol{v}, l) = 1) & \end{cases}$$

holds. See Figure 10.

From above discussion in Cases 1 to 3, if all $\rho(\boldsymbol{u}, \boldsymbol{v}, j)$ $(0 \leq j \leq \tilde{n})$ are given, calculation of $\Delta\rho = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) - \rho(\boldsymbol{u}, \boldsymbol{v})$ for all $\boldsymbol{u}^{(k)}$ $(0 \leq k \leq n-1)$ can be done in $O(n)$ time. Therefore, calculation of $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v})$ for all $\boldsymbol{u}^{(k)}$ $(0 \leq k \leq n-1)$ can be also done in $O(n)$ time. □

Figure 11 shows the algorithm that calculates an array $d[k]$ $(0 \leq k \leq n-1)$ with $d[k] = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) - \rho(\boldsymbol{u}, \boldsymbol{v})$ for vertices $\boldsymbol{u}$ and $\boldsymbol{v}$ in $CQ_n$. Note that $f_1$ and $f_2$ represent $\rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) - \rho(\boldsymbol{u}, \boldsymbol{v})$ in Case 1c and Case 2c, respectively. Also, note that because $r[\bar{k}] = \sum_{j=\bar{k}+1}^{\tilde{n}} \rho(\boldsymbol{u}, \boldsymbol{v}, j)$, it is possible to judge whether $(u'_{2\bar{k}+1}, u'_{2\bar{k}}) \approx (v_{2\bar{k}+1}, v_{2\bar{k}})$ or not in $O(1)$ time by checking if $r[\bar{k}]$ is odd or even.

For a source vertex $\boldsymbol{u} = (0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1)$ and a destination vertex $\boldsymbol{v} = (1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0)$ in $CQ_{12}$, we give an example of classification of neighboring vertices of the source vertex and intermediate vertices for sending a message from $\boldsymbol{u}$ to $\boldsymbol{v}$. Our algorithm classifies the neighboring vertices of $\boldsymbol{u}$, $N(\boldsymbol{u})$, into $N_{-1}(\boldsymbol{u}, \boldsymbol{v}) = \{(0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 1), (0, 1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1), (0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 1), (0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0)\}$, $N_0(\boldsymbol{u}, \boldsymbol{v}) = \{(1, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1), (0, 0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1), (0, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, 1)\}$, and $N_{+1}(\boldsymbol{u}, \boldsymbol{v}) = \{(0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1), (0, 1, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1), (0, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1)\}$. See Figure 12. The six vertices in $N_{-1}(\boldsymbol{u}, \boldsymbol{v})$ are on the shortest paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. That is, the distance from each vertiex in $N_{-1}(\boldsymbol{u}, \boldsymbol{v})$ is $d(\boldsymbol{u}, \boldsymbol{v}) - 1$. The three vertices in $N_0(\boldsymbol{u}, \boldsymbol{v})$ are on the sidetrack paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. That is, the distance from each vertiex in $N_0(\boldsymbol{u}, \boldsymbol{v})$ is $d(\boldsymbol{u}, \boldsymbol{v})$. The three vertices in $N_{+1}(\boldsymbol{u}, \boldsymbol{v})$ are on the backtrack paths from $\boldsymbol{u}$ to $\boldsymbol{v}$. That is, the distance from each vertiex in $N_{+1}(\boldsymbol{u}, \boldsymbol{v})$ is $d(\boldsymbol{u}, \boldsymbol{v}) + 1$. Among $N_{-1}(\boldsymbol{u}, \boldsymbol{v})$, we pick up the vertex $\boldsymbol{u}_1 = (0, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1)$ to forward the message. Then, $N(\boldsymbol{u}_1)$ is classified into three subsets: $N_{-1}(\boldsymbol{u}_1, \boldsymbol{v}) = \{(1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1)\}$, $N_0(\boldsymbol{u}_1, \boldsymbol{v}) = \{(0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1)\}$, and $N_{+1}(\boldsymbol{u}_1, \boldsymbol{v}) = \{(0, 1, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1), (0, 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1), (0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 0, 1), (0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1), (0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1), (0, 1, 1, 0, 0, 1, 0, 1, 1, 0,$

```
procedure class(u, v, ρ(u,v,j) (0 ≤ j ≤ ñ))
begin
  f1 := 0;
  f2 := -1;
  ħ := ⌊max{h | u_h ≠ v_h}/2⌋;
  r[ñ] := ρ(u,v,ñ);
  for i := ñ - 1 to 0 step -1 do
    r[i] := r[i+1] + ρ(u,v,i+1);
  for k := 0 to n - 1 do begin
    k̄ := ⌊k/2⌋;
    if k̄ < ħ then begin
      if ρ(u,v,k̄) = 0 then
        d[k] := 1
      else if (u'_{2k̄+1}, u'_{2k̄}) ≈ (v_{2k̄+1}, v_{2k̄}) then
        d[k] := -1
      else d[k] := f1;
      if k is even and u_k = v_k = 1 then
        if ρ(u,v,k̄) = 0 then
          f1 := 1
        else f1 := -1;
      if k is even and ρ(u,v,k̄) > 0 then
        if v_{k+1} ⊕ v_k = u_{k+1} then
          f2 := f1
        else f1 := -1 end
    else if k̄ = ħ then begin
      if ρ(u,v,k̄) = 2 then
        d[k] := -1
      else if ρ(u,v,k̄) = 1 and u_k = v_k then
        d[k] := 1
      else d[k] := f2 end
    else if ρ(u,v,ħ) = 1 then
      d[k] := 1
    else d[k] := f1
  end;
  return d
end
```

Fig. 11: Algorithm for calculation of $d[k] = \rho(\boldsymbol{u}^{(k)}, \boldsymbol{v}) - \rho(\boldsymbol{u}, \boldsymbol{v})$.

$1, 1), (0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 0)\}$. Among $N_{-1}(\boldsymbol{u}_1, \boldsymbol{v})$, we pick up the vertex $\boldsymbol{u}_2 = (1, 1, 1, 1, 0, 1, 0, 1, 1, 1, 0, 1)$ to forward the message. Then, $N(\boldsymbol{u}_2)$ is classified into three subsets: $N_{-1}(\boldsymbol{u}_2, \boldsymbol{v}) = \{(1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1)\}$, $N_0(\boldsymbol{u}_2, \boldsymbol{v}) = \emptyset$, and $N_{+1}(\boldsymbol{u}_2, \boldsymbol{v}) = N(\boldsymbol{u}_2) \setminus N_{-1}(\boldsymbol{u}_2, \boldsymbol{v})$. Since $N_{-1}(\boldsymbol{u}_2, \boldsymbol{v})$ is a singleton set, we select $\boldsymbol{u}_3 = (1, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1)$ to forward the message. Then, $N(\boldsymbol{u}_3)$ is classified into three subsets: $N_{-1}(\boldsymbol{u}_3, \boldsymbol{v}) = \{(1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1)\}$, $N_0(\boldsymbol{u}_3, \boldsymbol{v}) = \emptyset$, and $N_{+1}(\boldsymbol{u}_3, \boldsymbol{v}) = N(\boldsymbol{u}_3) \setminus N_{-1}(\boldsymbol{u}_3, \boldsymbol{v})$. Again, since $N_{-1}(\boldsymbol{u}_3, \boldsymbol{v})$ is a singleton set, we select $\boldsymbol{u}_4 = (1, 1, 1, 1, 1, 1, 0, 1, 0, 1, 0, 1)$ to forward the message. Then, $\boldsymbol{u}_4$ and $\boldsymbol{v}$ are adjacent, and the message is sent to $\boldsymbol{v}$. Figure 13 illustrates the construction of this path.

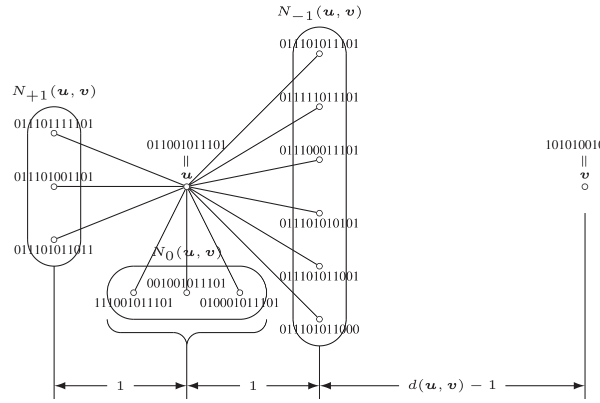


Fig. 12: Classification of neighboring vertices of $\boldsymbol{u}$.

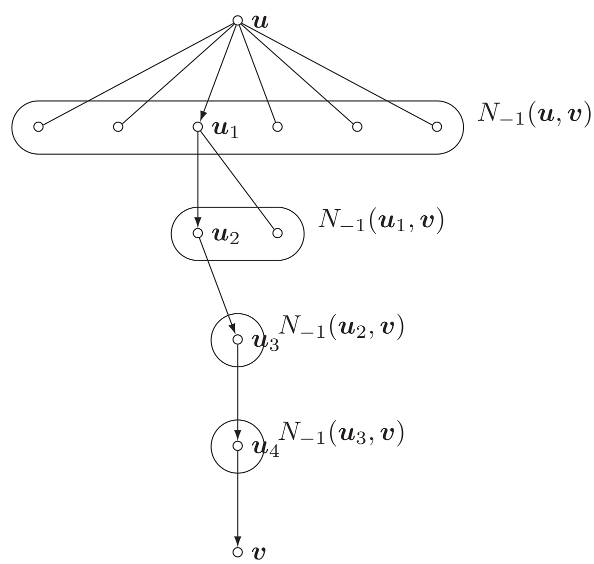


Fig. 13: An example of path construction by our method.

# 4. Conclusion and Future Works

In this paper, we have proposed an algorithm for the shortest-path routing in an $n$-dimensional crossed cube $CQ_n$. For a current vertex $\boldsymbol{c}$ and a destination vertex $\boldsymbol{d}$, the algorithm classifies the neighboring vertices of $\boldsymbol{c}$ into three subsets: $N_{-1}(\boldsymbol{c}, \boldsymbol{d}) = \{\boldsymbol{n} \mid \boldsymbol{n} \in N(\boldsymbol{c}), d(\boldsymbol{n}, \boldsymbol{d}) = d(\boldsymbol{c}, \boldsymbol{d}) - 1\}$, $N_0(\boldsymbol{c}, \boldsymbol{d}) = \{\boldsymbol{n} \mid \boldsymbol{n} \in N(\boldsymbol{c}), d(\boldsymbol{n}, \boldsymbol{d}) = d(\boldsymbol{c}, \boldsymbol{d})\}$, and $N_{+1}(\boldsymbol{c}, \boldsymbol{d})\{\boldsymbol{n} \mid \boldsymbol{n} \in N(\boldsymbol{c}), d(\boldsymbol{n}, \boldsymbol{d}) = d(\boldsymbol{c}, \boldsymbol{d}) + 1\}$ in $O(n)$ time. By forwarding the message to one of the elements in the first subset $N_{-1}(\boldsymbol{c}, \boldsymbol{d})$, fully adaptive minimal routing is attained.

Let us assume that the network based on the crossed cube contains some faulty vertices and/or edges. Then, we can utilize the classification of the neighboring vertices of the current vertex $\boldsymbol{c}$ for message forwarding. That is, to forward the message from $\boldsymbol{c}$ to one its neighboring vertices, we can select it among the three subsets with the priority $N_{-1}(\boldsymbol{c}, \boldsymbol{d})$, $N_0(\boldsymbol{c}, \boldsymbol{d})$, and $N_{+1}(\boldsymbol{c}, \boldsymbol{d})$ in this order so that the path will be shorter. To evaluate this approach is one of the future works.

# Acknowledgements

This study was partly supported by a Grant-in-Aid for Scientific Research (C) of the Japan Society for the Promotion of Science under Grant No. 17K00093.


# References


[1] S. B. Akers, D. Harel, and B. Krishnamurthy, "The star graph: An attractive alternative to the n-cube," in *Proceedings of the International Conference on Parallel Processing*, 1987, pp. 393–400.

[2] S.-I. Yeh, C.-B. Yang, and H.-C. Chen, "Fault-tolerant routing on the star graph with safety vectors," in *Proceedings of the Sixth Annual International Symposium on Parallel Architectures, Algorithms, and Networks*, May 2002, pp. 301–306.

[3] P. F. Corbett, "Rotator graphs: An efficient topology for point-to-point multiprocessor networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 3, no. 5, pp. 622–626, May 1992.

[4] K. Kaneko and Y. Suzuki, "An algorithm for node-to-set disjoint paths problem in rotator graphs," *IEICE Transactions on Information and Systems*, vol. E84-D, no. 9, pp. 1155–1163, Sept. 2001.

[5] H. Dweighter, "Elementary problem e2569," *The Americal Mathematical Monthly*, vol. 82, no. 1, p. 1010, 1975.

[6] S. Asai, Y. Kounoike, Y. Shinano, and K. Kaneko, "Computing the diameter of 17-pancake graph using a PC cluster," in *Euro-Par*, ser. Lecture Notes in Computer Science, W. E. Nagel, W. V. Walter, and W. Lehner, Eds., vol. 4128. Springer, 2006, pp. 1114–1124.

[7] L. Gardner, Z. Millter, D. Pritikin, and I. H. Sudborough, "Embedding hypercubes into pancake, cycle prefix and substring reversal networks," in *Proceedings of the 28th Annual Hawaii International Conference on System Sciences*, Jan. 1995, pp. 537–545.

[8] Y. Kikuchi and T. Araki, "Edge-bipancyclicity and edge-fault-tolerant bipancyclicity of bubble-sort graphs," *Information Processing Letters*, vol. 100, no. 2, pp. 52–59, Oct. 2006.

[9] Y. Suzuki and K. Kaneko, "The container problem in bubble-sort graphs," *IEICE Transactions on Information and Systems*, vol. E91-D, no. 4, pp. 1003–1009, Apr. 2008.

[10] K. Efe, "The crossed cube architecture for parallel computing," *IEEE Transactions on Parallel and Distributed Systems*, vol. 3, no. 5, pp. 513–524, Sept. 1992.

[11] P. Cull and S. M. Larson, "The möbius cubes," *IEEE Transactions on Computers*, vol. 44, no. 5, pp. 647–659, May 1995.

[12] J. Fan, "Hamilton-connectivity and cycle-embedding of the möbius cubes," *Information Processing Letters*, vol. 82, no. 2, pp. 113–117, Apr. 2002.

[13] D. Kocík and K. Kaneko, "Node-to-node disjoint paths problem in a möbius cube," *IEICE Transactions on Information and Systems*, vol. E100-D, no. 8, pp. 1837–1843, Aug. 2017.

[14] S. Abraham and K. Padmanabhan, "An analysis of the twisted cube topology," in *Proceedings of the International Conference on Parallel Processing*, vol. 1. Pennsylvania State Press, Aug. 1989, pp. 116–120.

[15] X. Yang, D. J. Evans, and G. M. Megson, "The locally twisted cubes," *International Journal of Computer Mathematics*, vol. 82, no. 4, pp. 401–413, Apr. 2005.

[16] W. J. Zhou, J. X. Fan, X. H. Jia, and S. K. Zhang, "The spined cube: a new hypercube variant with smaller diameter," *Information Processing Letters*, vol. 111, no. 12, pp. 561–567, June 2011.

[17] G.-L. Cheng, Q. Zhu, and X.-K. Wang, "On the reliability and fault tolerance of spined cubes," in *Proceedings of the International Conference on Wavelet Analysis and Pattern Recognition*, July 2012, pp. 313–316.

[18] Y. Li and S. Peng, "Dual-cube: A new interconnection network for high-performance computer clusters," in *Proceedings of the International Computer Symposium*, Dec. 2000, pp. 51–57.

[19] J. Arai and Y. Li, "Disjoint-path routing on hierarchical dual-nets," *International Journal of Networking and Computing*, vol. 4, no. 2, pp. 260–278, July 2014.

[20] J. Park, N. Seki, and K. Kaneko, "Stochastic fault-tolerant routing in dual-cubes," *IEICE Transactions on Information and Systems*, vol. E100-D, no. 8, pp. 1920–1921, July 2017.

[21] C. L. Seitz, "The cosmic cube," *Communications of the ACM*, vol. 28, no. 1, pp. 22–33, Jan. 1985.

[22] C.-P. Chang, T.-Y. Sung, and L.-H. Hsu, "A new shortest path routing algorithm and embedding cycles of crossed cube," in *Proceedings of the Third International Symposium on Parallel Architectures, Algorithms, and Networks*, Dec. 1997, pp. 125–131.

[23] ——, "Edge congestion and topological properties of crossed cubes," *IEEE Transactions on Parallel and Distributed Systems*, vol. 11, no. 1, pp. 64–80, Jan. 2000.

[24] H.-C. Chen, T.-L. Kung, and L.-H. Hsu, "An augmented pancyclicity problem of crossed cubes," *The Computer Journal*, vol. 61, no. 1, pp. 54–62, Jan. 2018.

[25] J. Fan, X. Jia, and X. Lin, "Complete path embeddings in crossed cubes," *Information Sciences*, vol. 176, no. 22, pp. 3332–3346, Nov. 2006.

[26] J. Fan, X. Lin, and X. Jia, "Node-pancyclicity and edge-pancyclicity of crossed cubes," *Information Processing Letters*, vol. 93, no. 3, pp. 133–138, Feb. 2005.

[27] ——, "Optimal path embedding in crossed cubes," *IEEE Transactions on Parallel and Distributed Systems*, vol. 16, no. 12, pp. 1190–1200, Dec. 2005.

[28] J.-S. Fu, H.-S. Hung, and G.-H. Chen, "Embedding fault-free cycles in crossed cubes with conditional link faults," *The Journal of Supercomputing*, vol. 49, no. 2, pp. 219–233, Aug. 2009.

[29] W.-T. Huang, Y.-C. Chuang, J. J.-M. Tan, and L.-H. Hsu, "On the fault-tolerant hamiltonicity of faulty crossed cubes," *IEICE Transactions on Fundamentals*, vol. E85-A, no. 6, pp. 1359–1370, June 2002.

[30] P. Kulasinghe and S. Bettayeb, "Embedding binary trees into crossed cubes," *IEEE Transactions on Computers*, vol. 44, no. 7, pp. 923–929, July 1995.

[31] P. D. Kulasinghe, "Connectivity of the crossed cube," *Information Processing Letters*, vol. 61, no. 4, pp. 221–226, Feb. 1997.

[32] P.-L. Lai and H.-C. Hsu, "Constructing the nearly shortest path in crossed cubes," *Information Sciences*, vol. 179, no. 14, pp. 2487–2493, June 2009.

[33] J.-M. Xu, M. Ma, and M. Lü, "Paths in möbius cubes and crossed cubes," *Information Processing Letters*, vol. 97, no. 3, pp. 94–97, Feb. 2006.

[34] M.-C. Yang, T.-K. Li, J. J. M. Tan, and L.-H. Hsu, "Fault-tolerant cycle-embedding of crossed cubes," *Information Processing Letters*, vol. 88, no. 4, pp. 149–154, Nov. 2003.

[35] X. Yang and G. M. Megson, "On the double-vertex-cycle-connectivity of crossed cubes," *Parallel Algorithms and Applications*, vol. 19, no. 1, pp. 11–17, Aug. 2004.

[36] Y.-H. Zhang, W. Hao, and T. Xiang, "Independent spanning trees in crossed cubes," *Information Processing Letters*, vol. 113, no. 18, pp. 653–658, Sept. 2013.

[37] J. Duato, S. Yalamanchili, and L. Ni, *Interconnection Networks: An Engineering Approach*. Los Alamitos, CA, USA: IEEE Computer Society Press, 1997.