On Operator-splitting Approach for Plasma 3-T Radiation Diffusion in Two and Three Dimensions

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Abstract—Operator-splitting approaches are often used for plasma 3-T radiation diffusion equations, since many linear solvers could not effectively solve the coupled linear system resulted from plasma 3-T radiation diffusion equations. An approach of operator-splitting is proposed for numerically solving plasma 3-T radiation diffusion equations. To resolve sub-cell structure in mixed cells, interface reconstruction is implemented within any mixed cell. Therefore, the system of 3-T radiation diffusion equations is solved on two- and three-dimensional polyhedral meshes. The focus of this paper is on the coupling between radiation and material, the treatment of nonlinearity in the equations, and the discontinuity across cell interfaces in material properties. The discontinuity of material properties between different materials is correctly treated based on the governing physics principle for general polyhedral meshes. The treatment is exact for arbitrarily strong discontinuity. The features of the resulting scheme are demonstrated through comparing with more accurate methods and some existing operator-splitting methods for numerical examples. It is demonstrated that the proposed approach is much better than the existing splitting approaches and close to unsplit method.

Keywords: diffusion, radiation, 3-T, coupling, interface, implicit

1. Introduction

Solving plasma 3-T radiation diffusion equations is often a critical step in numerical simulations of multi-physics. Many of these simulations involve mixed cells. Within a mixed cell there are more than one material. The existence of mixed cells is due to either initial distribution of material or other physics packages, such as hydrodynamics and advection.

For plasma 3-T diffusion equations to resolve the sub-cell structure within mixed cells are important for many problems. The importance is due to three facts. First, materials within a mixed cell are not necessarily in equilibrium with each other. The use of a single value for temperature within a mixed cell is inappropriate. Second, for real physics and engineering problems, tables for equation of state (EOS) are needed for diffusion coefficients, but EOS tables are unavailable for a mixture of materials. An averaging procedure of EOS tables cannot describe the mixture well. Third, an operator-splitting technique is used in many multi-physics codes, in which the result of one physics package is used as an input of a subsequent physics package. A subsequent physics could be temperature-sensitive, i.e., the result of subsequent physics sensitively depends on temperature of each material obtained from radiation diffusion solvers. If temperatures and material properties within a mixed cell are homogenized through some weighted average, after the homogenized temperature is updated through plasma 3-T radiation diffusion equations there is no physics principle available to separate the temperatures of different materials.

To resolve the sub-cell structure, a mixed cell is often decomposed to a set of sub-cells through interface reconstruction. The sub-cells generated through the reconstruction are general polygons in two dimensions and general polyhedrons in three dimensions. Therefore, the operator-splitting approach to be proposed here is for general polyhedral meshes.

For problems with multi-materials of dramatically different properties, a correct treatment for the discontinuity of material properties is important [1,2]. One approach for the calculation of flux near material interfaces on unstructured meshes is to use mathematical approximations, for example, some weighted average of two adjacent materials. This approach would introduce numerical errors when thermal properties of two materials are very different. In this paper, we will give a general formulation to calculate the effective diffusion coefficients of radiation, electrons, and ions at an interface between two different materials within the framework of polyhedral meshes. Another important aspect in 3-T radiation diffusion equations is the treatment of the nonlinearity in the equations.

There are many investigations on numerical methods for diffusion equations. Some are for single diffusion equation, some for coupled systems, some for structured meshes, and some for single material. Very few papers discuss operator-splitting approaches. In practice, it is often to take some straightforward approach for the 3-T diffusion equations, but some of them are obviously wrong for some problems even for very small time steps. One of typical operator-splitting methods is to separate the diffusion process from the interaction between radiation and material, and another is to separate each diffusion process from each other with an inappropriate treatment of the nonlinearity in the interaction between radiation and electrons. Both approaches result in serious errors in simulation results for some problems.

In this paper, we will propose an operator-splitting approach for 3-T radiation diffusion equations. The focus of
the development is on the coupling between radiation and material, the treatment of nonlinearity in the equations, and the discontinuity across cell interfaces in material properties. The discontinuity of material properties between different materials is correctly treated based on the governing physics principle for general polyhedral meshes. The plan of the paper is as follows. The second section is for 3-T radiation diffusion equations to be solved. In the third section, we will briefly describe a numerical scheme for 3-T radiation diffusion equations on general polyhedral meshes that is unsplit and fully nonlinear for the nonlinearity in the 3-T diffusion equations. After that we will present our operator-splitting algorithm for the 3-T equations. Numerical examples are given in Section 4 to demonstrate the features of the algorithm developed in this paper. In this section, we will also present comparison in simulation results among the proposed operator-split algorithm and the fully nonlinear and unsplit algorithm, and the two existing operator-splitting algorithms. Following numerical examples is the conclusion of this paper.

2. Plasma 3-T Radiation Diffusion Equations

In a system of plasma with radiation, when the time scale for equilibration of photons, electrons, and ions is much shorter than the time scale of the interaction between them, photons, electrons, and ions are each in a state of local thermodynamic equilibrium (LTE). Photons’ energy is given by a Planck distribution, and those for electrons and ions are Maxwellian. Their associated temperatures, \( T_r, T_e, \) and \( T_p \), are not necessarily in equilibrium.

Photons are described by an angle- and frequency-dependent equation of radiation transfer for radiation intensity. Expanding the intensity function in spherical harmonics, retaining the first two terms, and using Ficks’s law, we get the frequency-dependent diffusion equation for photons. Further integrating the equation over frequency gives the diffusion equation of the energy of photons. Together with the electron and ion energy transport equations, the set of 3-T radiation diffusion equations is

\[
\begin{align*}
\alpha \frac{\partial \phi}{\partial t} &= -\nabla \cdot \mathbf{F}_r + S_r, \quad (1) \\
C_{ve} \frac{\partial T_e}{\partial t} &= -\nabla \cdot \mathbf{F}_e - S_e + S_c, \quad (2) \\
C_{vp} \frac{\partial T_p}{\partial t} &= -\nabla \cdot \mathbf{F}_p - S_e. \quad (3)
\end{align*}
\]

Here \( \phi \equiv T_r^4 \), \( \alpha \) is the radiation constant, \( C_{ve} \) and \( C_{vp} \) are heat capacities of electrons and ions, and they are related with material mass density \( \rho \) and specific heat capacities \( c_{ve} \) and \( c_{vp} \) through \( C_{ve} \equiv \rho c_{ve} \) and \( C_{vp} \equiv \rho c_{vp} \). \( \mathbf{F}_r, \mathbf{F}_e, \) and \( \mathbf{F}_p \) are energy fluxes of radiation, electrons, and ions respectively, and they are defined as

\[
\begin{align*}
\mathbf{F}_r &= -\sigma_r \nabla \phi, \quad (4) \\
\mathbf{F}_e &= -\sigma_e \nabla T_e, \quad (5) \\
\mathbf{F}_p &= -\sigma_p \nabla T_p. \quad (6)
\end{align*}
\]

Here \( \sigma_r, \sigma_e, \) and \( \sigma_p \) are diffusion coefficients of radiation energy and temperatures of electrons and ions. \( \sigma_r \) is inversely proportional to Rosseland opacity that includes atomic processes such as Bremsstrahlung, photo-ionization, line-absorption, and Thompson scattering. The source terms in Eqs.(1-3) are defined as

\[
\begin{align*}
S_r &= a\rho \kappa_p (T_r^4 - T_e^4), \quad (7) \\
S_e &= C_{ve} \kappa_p (T_p - T_e). \quad (8)
\end{align*}
\]

In Eqs. (7,8), \( c \) is the light speed, \( \kappa_p \) Planck mean opacity, and \( \kappa_{pe} \) the coefficient for interaction between electrons and ions. The term, \( S_r \), in Eqs.(1,2) allows for the exchange of energy between radiation and electrons.

3. Numerical Schemes

In this section, we will first briefly describe a recently developed algorithm [2] for plasma 3-T radiation diffusion equations, and then propose an operator-splitting algorithm. Since the resulting meshes through reconstruction of interfaces between materials are unstructured, the numerical schemes to be presented in this section are for Eqs.(1-3) on a general polyhedral mesh in two and three dimensions. We will particularly focus on the effective diffusion coefficient with discontinuity of material properties, nonlinearity of the equations, and the coupling between radiation and material.

Referring to a cell in an unstructured mesh, the red cell as shown in Fig. 1, we denote \( N_i \) as a set of cells shown through the green cells in the figure, each of which neighbors the cell \( i \) with a non-vanishing interface. We will use \( \Delta t \) for the size of time step, \( \Delta V_i \) for the volume of cell \( i \), \( A_{ik} \) for the area of the interface between cell \( i \) and cell \( k \) as shown in the figure. We will use \( T_{ei}, T_{pi}, \) and \( T_{ri} \) for the temperatures of electrons, ions, and radiation at \( t = 0 \), \( T_{ei}, T_{pi}, \) and \( T_{ri} \) for the temperatures at \( t = \Delta t \). \( C_{vei} \) and \( C_{vpi} \) for heat capacities of electrons and ions in cell \( i \), \( \kappa_{pe} \) for the value of Planck mean opacity \( \kappa_p \) at cell \( i \) and \( \kappa_{pe} \) for the value of Planck mean opacity \( \kappa_p \) at cell \( i \). We will focus on Euler backward method in which the temperatures used in the calculation of fluxes \( \mathbf{F}_r, \mathbf{F}_e \) and \( \mathbf{F}_p \) are their values at \( t = \Delta t \).

3.1 Fully Nonlinear Difference Equations

Considering a cell in an unstructured mesh, the red cell in Fig. 1, we integrate Eq.(1-3) over the cell and one time step \( 0 < t < \Delta t \) with \( \Delta t \) the size of time step. After converting the volume integral into a surface integral over the enclosed surface of the cell for the fluxes and evaluate...
Fig. 1: Illustration of the procedure to calculate energy flux across the interface between cells $i$ and $k$. $l_i$ or $l_k$ is the vector from the center of cell $i$ or $k$ to the center of the interface. $T_i$ and $T_k$ are the temperatures at the centers of cells $i$ and $k$ respectively. $T_{ik}$ is the temperature at the center of the interface.

The fluxes through the temperatures at $t = \Delta t$, we could get the following set of nonlinear difference equations [3],

$$
[a_\gamma_i + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{r_{ik}} A_{ik}) + \tau_i^n] T_{ri}^n - \tau_i^n T_{ei}^n = a_\gamma_i T_{ri} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{r_{ik}} A_{ik} T_{rk}^n), \tag{9}
$$

$$
- \tau_i^n T_{ri}^n + |C_{vei} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{eik} A_{ik}) + \tau_i^n + C_{vei} \kappa_{pei} \Delta t T_{pi}^n - C_{vei} \kappa_{pei} \Delta t T_{pi}^n = C_{vei} T_{ei} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{ek} A_{ek} T_{ek}^n), \tag{10}
$$

$$
- C_{vei} \kappa_{pei} \Delta t T_{ei}^n + |C_{vp} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{pik} A_{ik}) + C_{vp} \kappa_{pei} \Delta t T_{pi}^n = C_{vp} T_{pi} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{pik} A_{ik} T_{pk}^n). \tag{11}
$$

In Eqs.(9,10), $\gamma_i^n$ and $\tau_i^n$ are defined as

$$
\gamma_i^n = \frac{[T_{ri}^n]}{[(T_{ri}^n)^2 + (T_{ei}^n)]},
$$

$$
\tau_i^n = \kappa_{pei} [(T_{ei}^n)^2 + (T_{ri}^n)^2] \Delta t. \tag{12}
$$

The coefficients $\sigma_{eik}$, $\sigma_{pik}$ and $\sigma_{r_{ik}}$ in Eqs.(9-11) are

$$
\sigma_{eik} \equiv \frac{\alpha_i \alpha_k \sigma_{eik}}{l_k \alpha_i \sigma_{ei} + l_i \alpha_k \sigma_{ek}}, \tag{14}
$$

$$
\sigma_{pik} \equiv \frac{\alpha_i \alpha_k \sigma_{pik}}{l_k \alpha_i \sigma_{pi} + l_i \alpha_k \sigma_{pk}}, \tag{15}
$$

$$
\sigma_{r_{ik}} \equiv \frac{\alpha_i \alpha_k \sigma_{r_{ik}}}{l_k \alpha_i \sigma_{ri} + l_i \alpha_k \sigma_{rk}} (T_{rk}^2 + T_{ri}^2)/(T_{rk} + T_{ri}). \tag{16}
$$

$\sigma_{r_{ik}}$ is the value $\sigma_{r_{ik}}$ evaluated at $T_{ri}^n$ and $T_{ei}^n$. The system of Eqs.(9-11) is nonlinear because $\sigma_{r_{ik}}$, $\gamma_i^n$, and $\tau_i^n$ depend on $T_{ei}^n$, $T_{pi}^n$ and $T_{ri}^n$.

The set of nonlinear difference equations, Eqs.(9-11), could be linearized, to get the following set of equations,

$$
[a_\gamma_i + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{r_{ik}} A_{ik}) + \tau_i^n] T_{ri}^n - \tau_i^n T_{ei}^n = a_\gamma_i T_{ri} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{r_{ik}} A_{ik} T_{rk}^n), \tag{17}
$$

$$
- \tau_i^n T_{ri}^n + |C_{vei} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{eik} A_{ik}) + \tau_i^n + C_{vei} \kappa_{pei} \Delta t T_{pi}^n - C_{vei} \kappa_{pei} \Delta t T_{pi}^n = C_{vei} T_{ei} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{ek} A_{ek} T_{ek}^n), \tag{18}
$$

$$
- C_{vei} \kappa_{pei} \Delta t T_{ei}^n + |C_{vp} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{pik} A_{ik}) + C_{vp} \kappa_{pei} \Delta t T_{pi}^n = C_{vp} T_{pi} + \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} (\sigma_{pik} A_{ik} T_{pk}^n). \tag{19}
$$

Here $\sigma_{r_{ik}}$, $\gamma_i$, $\tau_i$ are $\sigma_{r_{ik}}$, $\gamma_i^n$, $\tau_i^n$ but evaluated at the temperatures at $t = 0$, $T_{ei}$, $T_{pi}$ and $T_{ri}$.\[12pt]

### 3.2 Operator Split Approaches for Plasma 3-T Diffusion Equations

One could try to iteratively solve the nonlinear system, Eqs.(9-11), but most existing solvers only could deal with linear systems. Although existing linear solvers seem working for coupled linear systems, for example, the system of Eqs.(17-19), it is hard to find an iterative solver working for the coupled linear system, since many iterative solvers decouple a coupled system into three uncoupled systems at some preliminary step, for example, within coarse grids of multigrid methods, and then solve the coupled system at the fine grid. For strongly coupled systems, such as Eqs.(17-19), these iterative solvers often fail to converge.

In this subsection, we will propose an operator-split approach to approximately solve Eqs.(1-3). Only a single variable, $T_r$, or $T_e$, or $T_i$ is solved in each step of the operator-split approach. We first write the system of Eqs.(1-3) into three sets, which will be successively solved,

$$
\begin{cases}
\frac{\partial \phi}{\partial t} = -\nabla \cdot \mathbf{F}_r + S_r, \\
C_{vei} \frac{\partial T_r}{\partial t} = -S_r + f S_e, \\
C_{vp} \frac{\partial T_e}{\partial t} = -f S_e.
\end{cases} \tag{20}
$$

$$
\begin{cases}
C_{vei} \frac{\partial T_e}{\partial t} = -\nabla \cdot \mathbf{F}_e + f S_e, \\
C_{vp} \frac{\partial T_i}{\partial t} = -f S_e \tag{21}
\end{cases}
$$
Here \( f \equiv \frac{1}{2} \).

Therefore, to solve the original system, Eqs.(1-3), for one time step, we solve the system Eq.(20) first for the time step, which we called step 1, then we solve the system of Eqs.(21) called step 2 using the results obtained from step 1 as an initial condition, after that using the results of step 2, we finally solve Eqs.(22) called step 3. After these three steps, we will have updated the solution of Eqs.(1-3) one time step.

To solve Eq.(20) for one time step, considering a cell in an unstructured mesh, the red cell as shown in Fig.1, we integrate Eq.(20) over the cell and one time step \( 0 < t < \Delta t \) with \( \Delta t \) the size of time step. After converting the volume integral into a surface integral over the enclosed surface of the cell for the fluxes and approximately evaluating the fluxes at cell interfaces through the temperatures at \( t = \Delta t \), we get the following equation,

\[
a\phi_i^n = a\phi_i - \frac{\Delta t}{V_i} \sum_{k \in N_i} F_{rik} A_{ik} + S_{ri} \Delta t, \tag{23}
\]

Here \( S_{ri} \) (or \( S_{ei} \)) is the source terms defined as

\[
S_{ri} = \frac{1}{\Delta V_i} \int_0^{\Delta t} \int_{\Delta V_i} S_r(t, r) dV dt.
\]

The superscript \( n \) in Eq.(23) stands for the new time \( \Delta t \).
\( \phi_i^n, T_{ei}^n, \) and \( T_{pi}^n \) are the cell averages of \( \phi, T_e, \) and \( T_p \) at \( t = \Delta t \),

\[
\phi_i^n = \frac{1}{\Delta V_i} \int_{\Delta V_i} \phi_i(t, r) dV,
\]

\[
T_{ei}^n = \frac{1}{\Delta V_i} \int_{\Delta V_i} T_e(t, r) dV,
\]

\[
T_{pi}^n = \frac{1}{\Delta V_i} \int_{\Delta V_i} T_p(t, r) dV.
\]

\( \phi_i, T_{ei}, \) and \( T_{pi} \) are similarly defined at \( t = 0 \). \( F_{riik} \) in Eq.(23) is the time-averaged flux across the interface \( A_{ik} \) and is defined as

\[
F_{riik} = \frac{1}{A_{ik} \Delta t} \int_0^{\Delta t} \int_{A_{ik}} F_r(t, r) dA dt. \tag{26}
\]

The summation in Eq.(23) is over the set \( N_i \).

To approximately calculate the flux in Eq.(23), we evaluate the flux and the source term at the temperature at \( \Delta t \). Thus we have

\[
a\phi_i^n = a\phi_i - \frac{\Delta t}{\Delta V_i} \sum_{k \in N_i} F_{rik} A_{ik} + S_{ri} \Delta t. \tag{27}
\]

Here the superscript \( n \) stands for the evaluation at \( t = \Delta t \), \( F_{riik} \) is the flux evaluated at \( t = \Delta t \), defined as

\[
F_{riik} = \frac{1}{A_{ik}} \int_{A_{ik}} F_r(\Delta t, r) \cdot dA. \tag{30}
\]

In Eqs.(27-29), we have also evaluated the interaction between radiation and material and the interaction between electrons and ions at \( t = \Delta t \).

\[
S_{ri} = \frac{1}{\Delta V_i} \int_{\Delta V_i} S_r(\Delta t, r) dV. \tag{31}
\]

Now we move to the calculation of the fluxes from temperature needed in Eq.(27). As we stated before, there are discontinuities in diffusion coefficients across interfaces between cells, and thus spatial derivatives of temperature is discontinuous across the interfaces. Therefore, we could not use Taylor expansion across an interface to approximately evaluate flux if there is a strong discontinuity. Considering the interface between cell \( i \) and cell \( k \) in Fig.1, we use \( l_i \) to stand for the distance vector from the center of cell \( i \) to the center of the interface. \( l_i \) is the magnitude of \( l_i \), and we use \( l_k \) to denote the distance vector from the center of cell \( k \) to the center of the interface, \( l_k \) is its magnitude, \( \mathbf{n}_k \) is the unit vector along \( l_k \). We use \( \mathbf{n}_k \) to denote the normal direction of the interface. We define \( \alpha_i \) and \( \alpha_k \) are the projections of the directions \( l_i \) and \( l_k \) along the normal direction.

\[
\alpha_i \equiv \mathbf{n}_k \cdot (l_i/l_k), \quad \alpha_k \equiv -\mathbf{n}_k \cdot (l_k/l_k).
\]

The radiation energy flux across the interface [1,2]

\[
F_{rik} = -\sigma_{rik}(\phi_k - \phi_i), \tag{32}
\]

with definition

\[
\sigma_{rik} \equiv \frac{\alpha_i \alpha_k \sigma_{ri} \sigma_{rk}}{l_k \sigma_{ri} + \alpha_i \alpha_k \sigma_{rk}}. \tag{33}
\]

Here \( \sigma_{ri} \) and \( \sigma_{rk} \) are \( \sigma_r \) evaluated at cell \( i \) and cell \( k \) respectively.

We would like to point out that Eq.(16) are similar, but different. Equation (16) is the expression of the flux in term of the difference of \( T_r \), but Eq.(33) is the one in term of the difference in \( \phi \).

To eliminate \( T_{pi}^n \), From Eq.(29), and get

\[
T_{pi}^n = \frac{T_{pi} + f(\nuvei/\nuvpi) \kappa_{pi} \Delta t T_{ei}^n}{1 + f(\nuvei/\nuvpi) \kappa_{pi} \Delta t}. \tag{34}
\]

Also, from Eq.(28) we have

\[
T_{ei} - T_{ei}^n = -\frac{a \kappa_{pi} \Delta t}{\nuvei}[T_{ei}^n - (T_{ei}^n)^4 + f \kappa_{pi} \Delta t (T_{pi}^n - T_{ei}^n)]. \tag{35}
\]
For the nonlinear term, \([(T_{ei}^n)^4 - (T_{ri}^n)^4]\), we take the following approximation,

\[(T_{ei}^n)^4 - (T_{ri}^n)^4 \approx (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri})(T_{ei}^n - T_{ri}^n).\]  \(36\)

Through this approximation and Eqs.(34,35), we get

\[T_{ri}^n = \frac{1}{P}[T_{ei} + f\eta_i T_{pi} + \frac{ac_{pei} \Delta t}{c_{vei}} (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri})].\]  \(37\)

Here \(P\) and \(\eta_i\) are defined as

\[P_i = 1 + \frac{ac_{pei} \Delta t}{c_{vei}} (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri}),\]

\[\eta_i = \frac{\kappa_{pei} \Delta t}{1 + f(c_{vei}/c_{vp}) \kappa_{pei} \Delta t}.\]

Substituting Eq.(37) for \(T_{ri}^n\) into Eq.(36), and putting the result into the source term of Eq.(27), we have

\[S_{ri}^n \Delta t = \frac{1}{P} ac_{pei} \kappa_{pei} \Delta t (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri})\]

\[\{T_{ei} + f\eta_i T_{pi} - (1 + f\eta_i)T_{ri}\}.\]  \(38\)

To solve for \(T_{ri}^n\) in Eq.(27) with Eq.(37), we approximately \(T_{ri}^n\) in Eq.(38) as

\[T_{ri}^n \approx \frac{3}{4} T_{ri} + \frac{1}{4T_{ri}^3} \phi^n.\]  \(39\)

Thus, we rewrite Eq.(38) as

\[S_{ri}^n \Delta t = a(\chi_i - \xi_i \phi^n).\]  \(40\)

Here

\[\chi_i = \frac{1}{P} c_{vp} \kappa_{pei} \Delta t (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri})\]

\[\{T_{ei} + f\eta_i T_{pi} - \frac{3}{4}(1 + f\eta_i)T_{ri}\},\]

\[\xi_i = \frac{c_{pei} \kappa_{pei} \Delta t}{4P_i T_{ri}^3} (T_{ei}^2 + T_{ri}^2)(T_{ei} + T_{ri})(1 + f\eta_i).\]

With this approximation, we get the linear system for \(\phi_i^n\),

\[a(1 + \xi_i) + \frac{\Delta t}{\Delta V_i} (\sum_{k,\in N_i} \sigma_{rik}^h A_{ik}) \phi_i^n = a(\phi_i + \chi_i) + \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} (\sigma_{rik} A_{ik} \phi_k^n),\]  \(41\)

The linear system, Eq.(41), could be iteratively solved for \(\phi_i^n\) with \(i = 1, 2, ..., N\). For a given set of initial values of \(\phi_i, T_{ei}\) and \(T_{pi}\), we initial guess \(\phi_i^n\) as \(\phi_i\). Then for each \(i\), using most recent values of \(\phi_i^n\) on all the neighboring cells \(k\), we evaluate the right sides of Eq.(41) to update \(\phi_i^n\). After repeating this for each cell \(i\), we will have finished one iteration. Of course, we could use any existing linear solver to iteratively solve the system.

Thus, we have solved one step of the three in the operator-splitting approach. The solution of this step is considered the initial condition of the second step, Eq.(21). We apply the same approach with Eq.(20) to Eq.(21), we get

\[C_{vei} T_{ei}^n = C_{vei} T_{ei} - \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} F_{esk}^n A_{ik}\]

\[+ f C_{vei} \kappa_{pei} \Delta t (T_{pi}^n - T_{ei}^n).\]  \(42\)

Here \(F_{esk}^n\) is the flux evaluated at the interface between cells \(i\) and \(k\) at the time \(t = \Delta t\). For the flux to work for the discontinuity in diffusion coefficient, we have the following form of the flux,

\[F_{esk}^n = -\sigma_{esk}(T_{ek}^n - T_{ei}^n).\]  \(43\)

Here \(\sigma_{esk}\) is defined in Eq.(14). To eliminate \(T_{pi}^n\), we integrate the second equation of Eq.(21) over one time step and cell \(i\) and get

\[T_{pi}^n = \frac{1}{P_i} [T_{pi} + f(c_{vei}/c_{vp}) \kappa_{pei} \Delta t T_{ei}^n]

\[\frac{c_{vei}}{c_{vp}} (\sum_{k,\in N_i} \sigma_{esk} A_{ik} T_{ek}^n)\]  \(44\)

Putting Eqs.(44) into Eq.(42), we have the following linear system for \(T_{ei}^n\),

\[C_{vei} (1 + f\eta_i) T_{ei}^n + \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} \sigma_{esk} A_{ik} T_{ei}^n = C_{vei} (T_{ei} + \eta_i T_{pi}) + \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} (\sigma_{esk} A_{ik} T_{ek}^n).\]  \(45\)

Again, the set of Eq.(45) forms a linear system for \(T_{ei}^n, i = 1, 2, ..., N\). This linear system could be iteratively solved. This concludes the second step of the operator-splitting approach.

The third and last step of our operator-splitting approach is similar to the second step described above. From Eq.(22) and through integration over one time step and cell \(i\), we have

\[T_{ei}^n = \frac{1}{1 + \kappa_{pei} \Delta t} (T_{ei} + \kappa_{pei} \Delta t T_{pi}^n),\]  \(46\)

\[C_{vp} T_{pi}^n = C_{vp} T_{pe} + \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} F_{sk}^n A_{ik} S_{ei}^n \Delta t.\]  \(47\)

Here \(F_{sk}^n\) is the evaluation of the flux at the cell interface between cells \(i\) and \(k\) and at \(t = \Delta t\), which has the form,

\[F_{sk}^n = -\sigma_{sk} (T_{sk}^n - T_{pi}^n),\]  \(48\)

and \(\sigma_{sk}\) is defined in Eq.(15). Substituting Eqs.(46,48) into Eq.(47) we have the following difference equation for \(T_{pi}^n\),

\[C_{vp} (1 + c_{vei}/c_{vp}) T_{pi}^n + \frac{\Delta t}{\Delta V_i} (\sum_{k,\in N_i} \sigma_{sk} A_{ik}) T_{pi}^n = C_{vp} (T_{pi} + c_{vei}/c_{vp}) A_{pi} p(T_{ei}^n) + \frac{\Delta t}{\Delta V_i} \sum_{k,\in N_i} (\sigma_{sk} A_{ik} T_{ki}^n)\]  \(49\)
Again, this equation forms a linear system for unknowns $T^i_{pn}, i = 1, 2, ..., N$.

Therefore, in this approximation, the solution of the coupled plasma 3-T diffusion equations for one time step is decomposed into three independent linear systems, Eqs.(41,45,49), which are successively solved with the solution of a previous step as the initial condition of the subsequent step. This operator-splitting approach turns out to be useful and practical since many existing solvers don’t work for coupled linear or nonlinear systems.

At the end of this section, we would like to point out that some common operator-splitting approaches don’t work well for some problems. Here are two examples. The first one is to separate diffusion and the iteration between material and radiation. Only three independent interactions are involved in the first step of this approach, and only interactions are involved in the second step. In both step of this approach, temperature are fully implicitly solved. Another inappropriate operator-splitting approach that is currently used is very similar to what we proposed in this paper, but the following approximation is used to linearize the resulting difference equations,

$$
(T_e^n)^4 - (T_e^n)^4 \approx T_e^4 - T_e^4 + 4(T_e^3 T_n^4 - T_r^3 T_r^4). \tag{50}
$$

We will show the problems of these two approaches through numerical examples in the next section.

## 4. Numerical Examples

In this section we will provide numerical examples to show the features of the schemes described in Sections 3. The first example involves only diffusion of radiation temperature. There are no interaction between radiation and electron, and between electrons and ions. Initially, there are two materials separated by the circle as shown in the middle image of Fig.2. There is a thin layer between two materials. Within the thin layer cells are mixed, and interfaces between materials are linearly reconstructed. Radiation temperatures are 1.0 in the inner and 4.0 outer regions respectively. The middle image of the figure displays the diffusion coefficient. In the image at the right of Fig.2, we show the radiation temperature after one time step. In Fig.3 we show the results after 10 time steps. In the figure, we also show the results from the numerical scheme of completely nonlinear and fully coupled system, Eqs.(9-11).

The second example is a 3-T problem with the same geometry as the one in the last example. The feature of this problem includes large radiation diffusion coefficient, small diffusion coefficients of electrons and ions, large heat capacity, and strong coupling between radiation and electrons and between electrons and ions. Initially, the temperatures of radiation, electron, and ion are the same in the either inner or outer region, and they are 1.0 and 4.0. In Fig.4, we show the solutions after one time step. The temperatures at the bottom are from the scheme in this paper, and the temperatures at the top are obtained from the completely nonlinear and fully coupled scheme, Eqs.(9-11).

We would like to point out that some operator-splitting schemes don’t work for this problem. For example, the operator-splitting scheme, which completely separates diffusion and interaction,

$$
\begin{align*}
\frac{\partial T_e^4}{\partial t} &= -\nabla \cdot \mathbf{F}_r, \\
C_{ve} \frac{\partial T_e^4}{\partial t} &= -\nabla \cdot \mathbf{F}_e, \\
C_{vp} \frac{\partial T_e^4}{\partial t} &= -\nabla \cdot \mathbf{F}_p,
\end{align*}
\tag{51}
$$

results in the solution shown in Fig.5 after the time step. As the result, temperatures of electrons and ions almost don’t change. This is obviously incorrect.

For this problem, another operator-splitting scheme, which uses the linearization, Eq.(50), to remove the nonlinearity, produces the solution shown in Fig.6, in which temperature of electrons is much overheated. We would like to point out that even with much smaller time step, the operator-splitting approach that uses the approximation, Eq.(50), will produce this large overheat.
for many problems, most linear solvers failed to work for the coupled system derived from plasma 3-T equations, and some operator-splitting schemes don’t work either. In this paper, we have developed an operator-splitting scheme for plasma 3-T diffusion equations, which will give reasonable solutions for many problems some other operator-splitting scheme failed. The calculation of energy fluxes in the scheme is based on the conservation of energy across interfaces between cells. Therefore the scheme is accurate for the discontinuity of material properties across a cell interface. We have demonstrated the properties of the scheme through numerical examples.

Fig. 6: The numerical solutions after the one time step obtained from an inappropriately splitting scheme, which uses the linearization Eq.(50). The solution is obviously incorrect, because the temperatures of electrons and ions are much overheated.

5. Conclusions

Mixed cells are often encountered in numerical simulations. To resolve sub-cell structure of thermally non-equilibrium state within a mixed cell, we have implemented an approach to divide each mixed cell into a set of unstructured cells through linear interface reconstruction in two- and three-dimensions, and then to solve 3-T radiation diffusion equations on the reconstructed meshes. The reconstructed meshes include arbitrarily polygons in two dimensions and arbitrarily polyhedrons in three dimensions.

Typically, numerical schemes for 3-T radiation diffusion equations on the resulting polyhedral meshes are coupled systems of three temperatures, $T_r$, $T_e$ and $T_i$. Unfortunately,