Modified Bi-Cubic and Bi-Quintic B-Spline Basis Functions for Simulating a Thin Plate Structure

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Abstract—Historically, researchers have avoided using higher-degree polynomials when formulating a basis via the Finite Element Method, needing to employ polynomials that meet the minimal smoothness requirements of the state space. However, with the advancement of smart materials, the need for understanding, modifying, and implementing interpolating functions with higher order is of increasing importance, particularly when handling material discontinuities. In this work we present modified bi-cubic and bi-quintic basis functions for simulating a thin plate structure.

Index Terms—plate, structures, finite element method, B-splines

I. INTRODUCTION

With the Finite Element Method (FEM) researchers employ basis functions that meet prescribed smoothness requirements built into the state space [1]. Historically, researchers have avoided using higher-degree polynomials when formulating a basis via the FEM due to the oscillatory behavior associated with higher degree polynomials, needing to employ polynomials that meet the minimal smoothness requirements of the state space. However, with the advancement of smart materials, the need for understanding, modifying, and implementing interpolating functions with higher order is of increasing importance, particularly when handling material discontinuities. Furthermore, basis functions that do not naturally meet the boundary conditions associated with a given model have to be modified [2]. In this work we present modified bi-cubic and bi-quintic basis functions for simulating a thin plate structure.

II. BACKGROUND

The cubic B-splines [3] are given by (1) and the quintic B-splines [4] are given by (2). Both are shown graphically in Figure 1.

$$B_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-2}) - 3(x - x_{i-2}) \cdot h^2, & x \in [x_{i-1}, x_i] \\ -3(x - x_{i-1})^2, & x \in [x_{i-1}, x_{i+1}] \\ h^3 + 3h^2(x_{i+1} - x) - 3(x_{i+1} - x) \cdot h^2, & x \in [x_i, x_{i+1}] \\ -3(x_{i+1} - x)^2, & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+2} - x)^3, & x \in [x_{i+2}, x_{i+3}] \\ 0, & \text{otherwise} \end{cases}$$

\(Q_i(x) = \frac{1}{h^5} \begin{cases} (x - x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & x \in [x_{i-1}, x_i] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_m)^5, & x \in [x_{i}, x_{i+1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_m)^5 + 15(x - x_{i+1})^5, & x \in [x_i, x_{i+2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5 - 20(x - x_m)^5 + 15(x - x_{i+1})^5 - 20(x - x_{i+2})^5, & x \in [x_{i+2}, x_{i+3}] \\ 0, & \text{otherwise} \end{cases}$$

For a spline dependent upon the variable \(y\), refine the definitions given with a replacement on \(x\) with \(y\) and \(i\) with \(j\). A two dimensional spline is the product denoted as

$$B_{i,j}(x, y) = B_i(x)B_j(y)$$

\(Q_{i,j}(x, y) = Q_i(x)Q_j(y)$$

(3)

for the cubic and quintic splines respectively.

III. METHODOLOGY

A thin plate clamped at one edge, free at the remaining edges is considered as the initial model. We denote our clamped edge as \(\Gamma_1\), our free edges as \(\Gamma_2, \Gamma_3, \Gamma_4\). To reduce the model to a system of ODEs, the Finite Element Method is
The plate equation and boundary conditions are given below. The resulting ODE system is presented below.

\[ \rho \ddot{w} - \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial y \partial x} - \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{yx}}{\partial x \partial y} = 0 \]  

(4)

The equation is rewritten as a first order system of the form

\[ \dot{x}(t) = Ax(t), \]  

(5)

where

\[ x(t) = \begin{bmatrix} s(t) \\ \dot{s}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \]  

(6)

Here, \( D \) denotes the rigidity matrix. Linear combinations of the B-splines are employed to satisfy the essential boundary conditions on \( \Gamma_1 \). For simplicity, the plots are shown for \( n_x, n_y = 7 \). The combination for the bi-cubic splines are

\[ B_{0,j}(x,y) = B_{1,j}(x,y) - 2B_{0,j}(x,y) - 2B_{2,j}(x,y), \]  

\[ B_{i,j}(x,y) = B_{i+2,j}(x,y), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n. \]  

(8)

and the bi-quintic splines are

\[ Q_{0,j}(x,y) = Q_{2,j}(x,y) - 7Q_{0,j}(x,y) - 7Q_{4,j}(x,y) - Q_{1,j}(x,y) - Q_{3,j}(x,y)Q_{1,j}(x,y) = Q_{1+4,j}(x,y), \]  

\[ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n. \]  

(9)

The combinations are shown graphically in Figure 3.

\[ Q_{X,j}(x,y) = 7Q_{X,j}(x,y) + Q_{1,j}(x,y) + Q_{3,j}(x,y) \]  

\[ i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n. \]  

(10)

The plots are shown for \( \rho h J = 0 \) and \( \rho h J = 0 \). The displacement \( w(t, x, y) \) is generated by the linear combination of \( Q_j(x, y) \). The displacements are

\[ w(t, x, y) \approx \sum_{j=1}^{N} s_j(t)Q_j(x, y) \]  

where \( N = 7 \). The solution to the plate with the modified bi-cubic splines are shown in Figure 4.

\[ s_j(t) = \begin{bmatrix} s_1(t) \\ \vdots \\ s_N(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \]  

(6)