The Weakly Dimension-Balanced Pancyclic on $T_{m,n}$ for $m$ or $n$ Being Even and the Other Being Odd

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Abstract - Given a graph $G = (V, E)$ and a cycle $C$ on $G$, the edge set $E$ is divided into $k$ dimensions $E_1, E_2, \ldots, E_k$ for a positive integer $k$. The set of all $i$-dimensional edge of $C$, a subset of $E(C)$, is denoted as $E_i(C)$ for $1 \leq i \leq k$. If $|E_i(C)| - |E_j(C)| \leq 1 \ (|i - j| = 1, \text{ respectively for } 1 \leq i < j \leq k$), $C$ is called a dimension-balanced cycle (weakly dimension-balanced cycle, respectively). If $G$ contains a weakly dimension-balanced cycle of every (even, respectively) length between 3 (every even, respectively), $G$ is called weakly dimension-balanced bipancyclic (bipancyclic, respectively). Moreover, if $\alpha$ is 3 (4, respectively), $G$ is called weakly dimension-balanced pancyclic (bipancyclic, respectively). In this paper, we prove that for one of $m$ and $n$ is even, the other is odd, the toroidal mesh graph $T_{m,n}$ are weakly dimension-balanced bipancyclic when $m, n \geq 4$, and $T_{m,n}$ is $k(2m - 3)$-weakly dimension-balanced pancyclic for even $m$.

Keywords: Toroidal mesh graph, weakly dimension-balanced cycle, pancyclic

1 Introduction

A network is conveniently represented by a graph whose vertices represent the nodes (i.e., processors) of the network and whose edges represent the communication links of the network. This investigation uses the terms “network” and “graph”: “node” and “vertex”, and “link” and “edge”, interchangeably. Let $G = (V, E)$ be a connected graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set, $|V(G)|$ denotes the number of vertices and $|E(G)|$ denotes the number of edges, respectively. A Hamiltonian cycle of $G$ is a cycle that contains every vertex of $G$. $G$ is pancyclic (bipancyclic, resp.) if, for every (every even, resp.) $k$ in the range $3 \leq k \leq |V(G)|$, $G$ contains a cycle of length $k$. In interconnection graph, Hamiltonicity and Pancyclicity are two important properties to design the graph and have been widely discussed in the previous literature (see [1-7]).

Given a graph $G = (V, E)$, the edge set $E$ is divided into $k$ dimensions $E_1, E_2, \ldots, E_k$ for a positive integer $k$. For any cycle $C$ on $G$, the set of all $i$-dimensional edge of $C$, which is a subset of $E(C)$, is denoted as $E_i(C) = E(C) \cap E_i$. If $|E(C)| - |E_i(C)| \leq 1$ for any $1 \leq i < j \leq k$, $C$ is called a dimension-balanced cycle (DBC, for short). If a DBC of $G$ is also a Hamiltonian cycle, $C$ is called a dimension-balanced Hamiltonian cycle (Hamiltonian DBC, for short) of $G$ [6, 7]. When $G$ does not exist a dimension-balanced Hamiltonian cycle, we will try to investigate whether $G$ satisfies other criteria. So [8] extend the definitions of DBC and Hamiltonian DBC as: If $|E(C)| - |E_i(C)| \leq 3$ for any $1 \leq i < j \leq k$, a cycle $C$ is called a weakly dimension-balanced cycle (WDBC, for short). [9] defined some similar definitions for pancyclic and bipancyclic as follows. For some integer $\alpha \geq 3$, if $G$ contains a WDBC of every length between 3 ($\alpha$, resp.) to $|V(G)|$, $G$ is called weakly ($\alpha$-weakly, resp.) dimension-balanced pancyclic (WDB (\alpha-WDB, resp.) pancyclic, for short). If $G$ contains a WDBC of every even length between 4 ($\alpha$, resp.) to $|V(G)|$, $G$ is called weakly ($\alpha$-weakly, resp.) dimension-balanced bipancyclic (WDB (\alpha-WDB, resp.) bipancyclic, for short).

$T_{m,n}$ is called the toroidal mesh graph whose vertex set $V(T_{m,n}) = \{(x, y) \mid 0 \leq x \leq m - 1, 0 \leq y \leq n - 1\}$, and edge set $E(T_{m,n}) = \{(x_1, y_1) (x_2, y_2) \mid x_1 = x_2 \text{ and } y_1 = y_2 \equiv \pm 1 \ (\text{mod} \ n), \text{ or } y_1 = y_2 \text{ and } x_1 - x_2 \equiv \pm 1 \ (\text{mod} \ m)\}$. Figure 1 shows an example of $T_{5,5}$. For convenience, we define the set $\text{bridges}_1 = \{(0, i)(m - 1, i) \mid 0 \leq i \leq n - 1\}$ and the set $\text{bridges}_2 = \{(j, 0)(j, n - 1) \mid 0 \leq j \leq m - 1\}$. The toroidal mesh graph is a famous interconnection network which have been pay more attention recently [7, 8, 9, 10, 11]. For the toroidal mesh graph $T_{m,n}$, we set $E_1 = \{(x_1, y_1) (x_2, y_2) \mid y_1 = y_2 \text{ and } x_1 - x_2 \equiv \pm 1 \ (\text{mod} \ m)\}$, and $E_2 = \{(x_1, y_1) (x_2, y_2) \mid x_1 = x_2 \text{ and } y_1 - y_2 \equiv \pm 1 \ (\text{mod} \ n)\}$ intuitively.

In this paper, we prove that $T_{m,n}$ contains a weakly dimension-balanced cycle with length $(4k + 2)$ for any integer $1 \leq k \leq \lfloor (mn - 2) / 4 \rfloor$, for one of $m$ and $n$ is even, the other is odd at first. Then, we investigate lemmas from previous works, $T_{m,n}$ is WDB bipancyclic when $m, n \geq 4$, and $T_{m,n}$ is $(2m - 3)$-WDB pancyclic for even $m$ is obtained. The rest of this paper is arranged as follows. Section 2 discusses preliminary of the main problem. In Section 3, we show the proof for the main result. Finally, in Section 4, we conclude the results of this paper.
2 Preliminary

This section gives some definitions and theorems that are used through this paper at first. In a toroidal mesh graph $T_{m,n}$, $R^1$ and $R^2$ are two induced subgraphs of $T_{m,n}$ with the vertex set $V(R^1) = \{(i, y) : 0 \leq i \leq m - 1\}$ for any $0 \leq y \leq n - 1$, and $V(R^2) = \{(x, i) : 0 \leq i \leq n - 1\}$ for any $0 \leq x \leq m - 1$. Let $R^1_{i,j} = \langle (i, y), (i + 1, y), \ldots, (j, y)\rangle$ be a route with vertices $(i, y)$ to $(j, y)$ for $0 \leq i \leq j < m$; $R^2_{i,j} = \langle (x, i), (x, i + 1), \ldots, (x, j)\rangle$ be a route with vertices $(x, i)$ to $(x, j)$ for $0 \leq i \leq j < n$. In 2017, Lai et al. studied the DB pancyclicity on $T_{m,n}$ for one of $m \geq 3$ and $n \geq 3$ is even, the other is odd [11]. We conclude their results in Table 1. In this table, without loss of generality, let $m$ be even and $n$ be odd.

<table>
<thead>
<tr>
<th>$m, n \geq 4$</th>
<th>$4k$-DBC</th>
<th>$(4k + 2)$-DBC</th>
<th>$(2k + 1)$-DBC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k = 1, 2, 3$ and $3 \leq k \leq m/2$</td>
<td>Yes, $1 \leq k \leq \lfloor mn/4 \rfloor$</td>
<td>Yes, $n - 1 \leq k \leq mn/2 - 1$; No, $1 \leq k &lt; n - 1$</td>
<td>No</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>$k = 2, 3, 4, 5, 7$ and $m - 1 \leq k \leq 3m/2 - 1$</td>
<td>Yes</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>$3 \leq k &lt; m/2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Lemma 1. If $m$ is even, $T_{m,3}$ does not contain any $4k$-WDBC for $3 < k < m/2$.

Proof. Assume that $C$ is a $4k$-WDBC on $T_{m,3}$, since there is no $4k$-DBC for $3 < k < m/2$, we let $|E(C)| = 2k + 1$ (or $2k - 1$, respectively) and $|E(C)| = 2k - 1$ (or $2k + 1$, respectively). By Property 1, we know if $m$ is even, then $|E(C)|$ must be even. So, if we let $|E(C)| = 2k + 1$ ($2k - 1$, respectively), $|E(C)|$ can only be odd. That is a contradiction. Hence, $T_{m,3}$ does not contain any $4k$-WDBC for $3 < k < m/2$.

Lemma 2. If $m$ is even, $T_{m,3}$ does not contain every $(4k + 2)$-WDBC for $4 < k < m/2 - 1$.

Proof. Assume that $C$ is a $(4k + 2)$-WDBC on $T_{m,3}$ for $4 < k < m/2 - 1$. Since there is no $(4k + 2)$-DBC for any integer $k$, and we know $|E(C)|$ is even by Property 1, so the only possible situation are $|E(C)| = 2k + 2$ ($< m$) and $|E(C)| = 2k$ ($|E(C)| = 2k$ ($< m$) and $|E(C)| = 2k + 2$, resp.). Therefore, $|E(C)|$ is even and $C$ contains even number of edges in bridges. And there is no edge in bridge be used in $C$. Let $x \in V(C)$ and $d(x)$ denote the degree of $x$, be the number of edges incident with $x$ in $C$. Then, let $Z_m = \{0, 1, \ldots, m - 1\}$ and $E_1(C) = \{(x_1, y)(x_2, y), (x_1, y)(x_2, y), (x_1, y)(x_2, y)\} \in E(C)\}$ for $0 \leq x \leq m - 1$. Let $R^1_{i,j} = \langle (i, y), (i + 1, y), \ldots, (j, y)\rangle$ be a route with vertices $(i, y)$ to $(j, y)$ for $0 \leq i \leq j < m$; $R^2_{i,j} = \langle (x, i), (x, i + 1), \ldots, (x, j)\rangle$ be a route with vertices $(x, i)$ to $(x, j)$ for $0 \leq i \leq j < n$. In 2017, Lai et al. studied the DB pancyclicity on $T_{m,n}$ for one of $m \geq 3$ and $n \geq 3$ is even, the other is odd [11]. We conclude their results in Table 1. In this table, without loss of generality, let $m$ be even and $n$ be odd.

In the following, we consider the “No” parts in Table 1 step by step and try to answer whether there exist WDB $k$-cycle for any $3 \leq l \leq mn$ in $T_{m,n}$ for one of $m, n$ is even and the other is odd.
even; \(|E_2(C)|\) is odd and \(C\) contains odd number of edges in bridges\(^2\), so let \(|E_1(C)| = k_1\) and \(|E_2(C)| = k_2\) where \(k_1 + k_2 = 2k + 1\), \(k_1\) be even, \(k_2\) be odd and \(|k_1 - k_2| = 3\). Note that in such assumption, \(k_1 = k + 3\) or \(k - 1\), both < \(m\), so that there is no edge in bridge\(^1\) be used in \(C\). This proof is similar to Lemma 2. Since cycle is a 2-regular graph, we know that for \(i \in Z_0\) and \(i' = (i + 1) \mod m, 6 \geq d_i((i', 0)) + d_i((i', 1)) + d_i((i', 2)) = |E_1(C)| + |E_2(C)| + 2|E_3(C)|\) is even. If \(|E_1(C)| + |E_3(C)| = 2\) then \(|E_2(C)| \leq 2\), this case only occurs when \(|i = m - 1\) or \(|i = k + 2\) - 1\). If \(|E_1(C)| + |E_3(C)| = 4\) then \(|E_2(C)| \leq 1\), this case only occurs when \(|i \leq (k + 2) - 1\). Thus \(k_2 = |E_2(C)| \leq 4\) and \(|k_2 - 1\)| - 1. If \(k_1 = k + 3\), then \(k_2\) and \(|k_1 - k_2| \leq 0\).

When \(k = 9\), \(2k + 1 = 19\), so that \(k_1 = 8\) and \(k_2 = 11\), a contradiction. When \(10 < k < m - 2\), \(21 < 2k + 1 = k_1 + k_2 \leq 21\), a contradiction. Therefore, \(T_{m,3}\) does not contain every \((2k + 1)\)-WDBC for \(m = 9\) or \(10 < k < m - 2\).

Note that a DBC is a WDBC by the definition. According to Lemma 3 and Table 1, we have the result that when \(k = 1, 6, 8, 10\) or \(m - 2\), it is possible to find \((2k + 1)\)-WDBC on \(T_{m,3}\) for \(m\) is even. Figure 3 shows the structure of \((2k + 1)\)-WDBC on \(T_{m,3}\) for \(k = 1, 6, 8, 10\), respectively. And Lemma 4 give a \((2k + 1)\)-WDBC on \(T_{m,3}\) for \(k = m - 2\).

**Fig. 3:** For \(m\) is even \(T_{m,3}\) contain \((2k + 1)\)-WDBC.

**Lemma 4.** If \(m \geq 6\) is even, \(T_{m,3}\) contains \((2k + 1)\)-WDBC for \(k = m - 2\).

**Proof.** For \(k = m - 2\), let \(|E_1(C)| = m = k + 2\), \(|E_2(C)| = k - 1 = m - 3\) and \(\beta_1 = (k - 1) - 3 = k - 4\). \(m - 6 \geq 0\). We can construct a \((2k + 1)\)-cycle \(C = \langle 0, 0 \rangle, (0, 2), (0, 1), (1, 1), (1, 2), (2, 2), (2, 1), \ldots, (\beta_1 - 2, 2), (\beta_1, 2), R_{\beta_1,m-1}, (m - 1, 2), R_{2,0}^{2m-1}, (m - 1, 0), (0, 0)\). Figure 4 shows the structure of \(C\). We have \(|E_1(C)| = m = k + 2\) and \(|E_2(C)| = 4 + \beta_1 + 1 = k - 1\). Then \(|E_1(C)| - |E_3(C)| = 3\). That is, \(C\) is a \((2k + 1)\)-WDBC for \(k = m - 2\).

**Fig. 4:** The construction of \(C\) on \(T_{m,3}\) for Lemma 4.

**Lemma 5.** For one of \(m\) and \(n\) is odd, the other is even, \(T_{m,n}\) exists \((2k + 1)\)-WDBC where \(k = m - 2\) for \(m > 3\) is odd, and \(n\) is even; or \(k = n - 2\) for \(n > 3\) is odd, and \(m\) is even.

**Proof.** Assume that \(C\) is a \((2k + 1)\)-WDBC on \(T_{m,n}\). Without loss of generality, let \(m\) be odd and \(n\) be even. By Property 1 and 2, we know \(|E_1(C)|\) is even; \(|E_1(C)|\) is odd and \(C\) contains odd number of edges in bridges\(^1\), so \(|E(C)|\) must \(\geq m\). Then \(|E_1(C)| = 2k + 1 - |E_3(C)| \leq 2k + 1 - m < 2(m - 2) + 1 - m = m - 3\), and \(|E_1(C)| - |E_3(C)| > 3\) is a contradiction. Therefore, \(T_{m,n}\) does not exist \((2k + 1)\)-WDBC, where \(1 \leq k < m - 2\) for \(m > 3\).

**Fig. 5:** The construction of \(C\) on \(T_{m,n}\) when \(\gamma \leq n - 1\) for Lemma 5.

**Fig. 6:** The construction of \(C\) on \(T_{m,n}\) when \(\gamma > n - 1\) for Lemma 5.
3 is odd, and \( n \) is even; or \( 1 \leq k < n - 2 \) for \( n > 3 \) is odd, and \( m \) is even.

## 3 Main results

In this section, for even integer \( m \geq 4 \) we prove that there exists a weakly dimension-balanced cycle whose length is \((4k + 2)\) for any integer \( m/2 - 1 \leq k \leq \lfloor (3m - 2)/4 \rfloor \) for \( T_{m,n} \), and \( 1 \leq k \leq \lfloor (mn - 2)/4 \rfloor \) for \( T_{m,n} \) where \( n \geq 5 \) is odd. We give Theorem 1 as the beginning for \( n = 3 \). Next, Lemma 7 and Theorem 2 present the general case.

**Theorem 1.** For \( m \geq 4 \) is even integer, \( T_{m,3} \) contains every \((4k + 2)\)-WDBC for \( m/2 - 1 \leq k \leq \lfloor (3m - 2)/4 \rfloor \).

**Proof.** When \( k = \lfloor (3m - 2)/4 \rfloor = (3m - 2)/4 \), \( T_{m,3} \) embedded a Hamiltonian WDBC, for any even integer \( m \) by [8]. Thus, we only need to discuss the cases: (a) \( m/2 - 1 \leq k \leq \lfloor (3m - 2)/4 \rfloor - 1 \) or (b) \( k = \lfloor (3m - 2)/4 \rfloor = (3m - 2)/4 \) for \( 3m \mod 4 = 0 \). Both implies \( 4k < 3m - 2 \). Since there is no \((4k + 2)\)-DBC in these case [11] and \( E_1(C) \) must be even by Property 1. We know that \( |E_1(C)| = |E_2(C)| = 2k, 2k + 2 \). In the following, we will construct a WDBC \( C \) on \( T_{m,3} \) with \( E_1(C) = 2k + 2 \) and \( E_2(C) = 2k \) for \( 2m + 2 < 4k < 4k < 3m \). Let \( m_1 = (2k + 2 - m)/2, \beta = ((2k - 4) \mod 4)/2 = (k - 2) \mod 2 = k \mod 2, n_1 = (2k - 4 - 2\beta)/4 = \lfloor k/2 \rfloor - 1 \). We construct a \((4k + 2)\)-cycle \( C \) shows in Figure 7. Obviously, \( |E_1(C)| = m + m_1 = 2k + 2 \) and \( |E_2(C)| = 4 + 2(n_1 + 2\beta) = 2k \). Hence \( C \) is a \((4k + 2)\)-WDBC.

Note that, by definition of \( \beta \), the only possible value of \( \beta \) is \( 0 \) or \( 1 \). We know that if \( 2n_1 + m_1 + 1 \geq m - 2 \) when \( \beta = 1 \), or \( 2n_1 + m_1 + 1 < m \) when \( \beta = 0 \), there will be wrong when constructing \( C \). Thus, we need to ensure \( 2n_1 + m_1 + 1 < m - 2 \) when \( \beta = 1 \), and \( 2n_1 + m_1 + 1 < m \) when \( \beta = 0 \). By definition, \( 2n_1 + m_1 + 1 = (2k - 4 - 2\beta + 2k + 2 - m)/2 + 1 = (4k - m - 2\beta - 2)/2 + 1 = (4k - m - 2\beta)/2 \). Therefore, \( 2n_1 + m_1 + 1 < m - 2 \) if \( \beta = 1; 2n_1 + m_1 + 1 < m - 1 < m \) if \( \beta = 0 \). Hence, \( C \) is a well-defined \((4k + 2)\)-WDBC.

![Fig. 7: The construction of \( C \) on \( T_{m,3} \) for Theorem 1.](image)

**Lemma 7.** For \( m \geq 4 \) is even integer, \( n \geq 5 \) is odd, \( T_{m,n} \) contains every \((4k + 2)\)-WDBC for \( 1 \leq k \leq \max\{m, n\} - 1 \).

**Proof.** Note that \( T_{m,n} = C_n \times C_m \). According to \( m, n, \) and \( k \), we divided this proof into two cases.

**Case 1.** \( 1 \leq k \leq \min\{m, n\} - 1 \)

If \( m > n \), construct a \((4k + 2)\)-cycle \( C = \langle(0, 0), R_{0,k}^{2k+1}, (0, k), R_{k,0}^{k+1}, (k + 1, k), R_{k,0}^{k+1}, (0, 0) \rangle \). Obviously, \( |E_1(C)| = 2k + 2 \) and \( |E_2(C)| = k + k = 2k \). Hence \( C \) is a \((4k + 2)\)-WDBC. If \( n > m \), construct a \((4k + 2)\)-cycle \( C = \langle(0, 0), R_{0,k}^{2k+1}, (0, k + 1), R_{k,0}^{k+1}, (k, 0), R_{k,0}^{k+1}, (0, 0) \rangle \). Obviously, \( |E_1(C)| = k + k = 2k \) and \( |E_2(C)| = 2k + 2 \). Hence \( C \) is a \((4k + 2)\)-WDBC.

**Case 2.** \( \min\{m, n\} \leq k \leq \max\{m, n\} - 1 \)

If \( m > n \), let \( \beta = ((2k - 2m + 4) \mod 2(n - 2))/2 = (k - n + 2) \mod (n - 2) \). Then construct a \((4k + 2)\)-cycle \( D_1 \) shows in Figure 8. Note that because \( m \geq 4 \) is even integer, \( n \geq 5 \) is odd, \( 2m_1 = 2\lfloor (k - n + 2)/(n - 2) \rfloor \leq k - 3 \), the structure of \( D_1 \) is well-defined. Besides, \( |E_1(D_1)| = k + k = 2k \) and \( |E_2(D_1)| = 2n - 2m_1(n + 2) + 2\beta = 2k + 2 \). That is, \( D_1 \) is a \((4k + 2)\)-WDBC.

If \( n > m \), let \( \alpha = ((2k - 2m + 4) \mod 2(n - 2))/2 = (k - m + 2) \mod (m - 2) \). Then construct a \((4k + 2)\)-cycle \( D_2 \). Figure 9 shows the structure of \( D_2 \). Note that because \( m \geq 4 \) is even integer, \( n \geq 5 \) is odd, \( 2m_1 = 2\lfloor (k - m + 2)/(n - 2) \rfloor \leq k - 3 \), the structure of \( D_2 \) is well-defined. Besides, \( |E_1(D_2)| = 2m - 2m_1(m - 2) + 2\alpha = 2k + 2 \) and \( |E_2(D_2)| = k + k = 2k \). That is, \( D_2 \) is a \((4k + 2)\)-WDBC.

![Fig. 8: The construction of \( D_1 \) for Lemma 7.](image)

![Fig. 9: The construction of \( D_2 \) on \( T_{m,n} \) for Lemma 7.](image)

**Theorem 2.** For \( m \geq 4 \) is even integer, \( n \geq 5 \) is odd, \( T_{m,n} \) contains every \((4k + 2)\)-WDBC for \( 1 \leq k \leq \lfloor (mn - 2)/4 \rfloor \).

**Proof.** According to Lemmas 7, when \( 1 \leq k \leq \max\{m, n\} - 1 \), \( T_{m,n} \) embeds every \((4k + 2)\)-WDBC for \( m \geq 4 \) is even integer, \( n \geq 5 \) is odd. In addition, when \( k = \lfloor (mn - 2)/4 \rfloor \) and \( mn \mod 4 = 2 \), \( T_{m,n} \) embedded a Hamiltonian WDBC, for any \( m, n \geq 3 \) by [8]. Thus, we only need to discuss that case: \( \max\{m, n\} \leq k \leq \lfloor (mn - 2)/4 \rfloor - 1 \) or \( k = \lfloor (mn - 2)/4 \rfloor \) for \( mn \mod 4 = 0 \). That can...
be rewrite as: \( \max\{m, n\} \leq k \leq \lceil (mn - 2)/4 \rceil - 1 \). Since there is no \((4k + 2)\)-DBC in these case [11] and \([E_1(C)], [E_2(C)]\) must be even by Property 1. We know that \([E_1(C)], [E_2(C)]\) = \([2k, 2k + 2]\).

In the following, we will construct a WDBC \(C\) on \(T_{m,n}\) with \([E_1(C)] = 2k + 2\) and \([E_2(C)] = 2k\). Let \(\alpha = (2k + 2 - m) \mod (n - 1)\), \(m = (2k + 2 - m - \alpha)/(n - 1)\) and \(\beta = ((2k - 2n + 2) \mod (2n - 2))/2 = (k - n + 1) \mod (n - 1) \equiv k \mod (n - 1)\), \(m = (2k - 2n + 2 - 2\beta)/(2n - 2) = [k/(n - 1)] - 1\). Note that \(4k < mn - 2\). According to the value of \(m\) and \(\alpha + 2\beta\), we separate this proof into three cases.

**Case 1.** \(m_1 > 0\) and \(\alpha_1 + 2\beta_1 < n - 1\).

At first, let \(m = m_1 - 1\), \(\alpha_1 = \alpha_1 + n - 1\), \(\gamma_1 = (\alpha_2 \mod 4)/2\) and \(\gamma_2 = (\alpha_2 - 2\gamma)/4\). If \(\gamma_1 = 0\), we construct a \(4k\)-cycle as follows: \(D_1 = (0, 0), R_{2,0,1}^{1,0}, (0, n - 1), R_{1,1}^{2,0}, (1, 0), (2, 0), R_{2,0,1}^{2,0}, (2, n - 1), ... (2n_1, n - 1), (2n_1 + 1, n - 1), R_{2,1}^{2,1,2,1}, (2n_1 + 1, n - 1), (2n_1 + 2 + \gamma_1 + 1, n - 2), R_{2,1}^{2,2,2,1}, (2n_1 + 1, n - 2), (2n_1 + 1, n - 3), R_{2,1}^{2,3,2,1}, (2n_1 + 2 + \gamma_1, n - 3), (2n + 1 + n - 1, 2n + 1 + n - 4), R_{2,1}^{2,4,2,1}, (2n + 1, n - 4), (2n + 1, n - 5), R_{2,1}^{2,5,2,1}, (2n + 1 + n - 5), (2n + 1 + n - 6), ... (2n_1 + 1, 0), R_{2,1}^{2,1,1,0,1}, (m - 2, 0), R_{0,0,1}^{3,0,0,1}, (m - 2, \beta_1), (m - 1, 0), R_{2,1}^{2,1,1,0,1}, (m - 1, 0), (0, 0))\). Figure 10 shows the construction of \(D_1\). By the construction of \(D_1\), we know that \([E_1(D_1) = m + m_1(n - 1) + 2\gamma_1 = m + (m - 1)(n - 1) + \alpha + n - 1 = 2k + 2\) and \([E_2(D_1) = 2n - 2 + 2m(n - 1) + 2\beta = 2k\). Hence, \(D_1\) is a \((4k + 2)\)-WDBC. Similarly, if \(\gamma_1 > 0\), we construct a \((4k + 2)\)-cycle \(D_2\) shows in Figure 11. Then \([E_1(D_2) = m + m_1(n - 1) + 4\gamma_1 = m + (m - 1)(n - 1) + (\alpha + n - 1) = 2k + 2\) and \([E_2(D_2) = 2n - 2 + 2n_1(n - 1) + 2\beta_1 = 2k\). So \(D_2\) is a \((4k + 2)\)-WDBC.

According to Figure 10 and Figure 11, note that \(\gamma_1 \leq 1\), we know that if \(2n_1 + m + 3 > m - 1\) or \(\beta_1 \geq n - 2\sigma_1 - 2\gamma_1\), there will be an overlap when constructing \(D_1\) and \(D_2\). Thus, we need to ensure \(2n_1 + m + 3 \leq m - 1\) and \(\beta_1 < n - 2\sigma_1 - 2\gamma_1\), \(\gamma_1 = 1\) and \(\beta_1 < n - 2\sigma_1\) separately. Because \(4k < mn - 2\), then \(2n_1 + m + 3 = 2n_1 + m + 2 = (4k - m - 2n + 4 - \alpha_1 - 2\beta_1)/(n - 1) + 2 = (4k - m - 2\beta)/(n - 1) < (mn - m - \alpha_1 - 2\beta)/(n - 1)\). Since \(0 \leq \alpha_1 + 2\beta_1\) and \(2n_1 + m + 2\) is an integer, \(2n_1 + m + 3 \leq m - (\alpha_1 + 2\beta_1)/(n - 1) \leq m\). So \(2n_1 + m + 3 \leq m - 1\). Next, we check the range of \(\beta\). Since \(\alpha_1 + 2\beta_1 < n - 1\) and \(\alpha_1 = 4\sigma_1 + 2\gamma_1 - n + 1\), \(2\sigma_1 + \gamma_1 + \beta_1 < n - 1\). Then \(\beta_1 < n - 2\sigma_1 - 1\). If \(\gamma_1 = 0\), \(\beta_1 < n - 2\sigma_1\). In summary, \(D_1\) and \(D_2\) are well-defined cycles.

**Case 2.** \(m_1 = 0\) or \(n - 1 \leq \alpha_1 + 2\beta_1 < n - 2\).

Let \(\gamma_2 = (\alpha_1 \mod 4)/2\) and \(\gamma_2 = (\alpha_1 - 2\gamma)/4\). According to \(\sigma_2\), we use \(m_1, \sigma_2\) and \(\gamma_2\) replace \(m_2, \sigma_1\) and \(\gamma_1\) in \(D_1\) or \(D_2\), then construct a cycle \(D_3\) (or \(D_4\), respectively) for \(\sigma_2 = 0\) (or \(\sigma_2 > 0\), respectively). Figure 12 shows the structure of \(D_3\) for \(\alpha_2 > 0\). By Case 1, \([E_1(D_3)] = [E_2(D_3)] = m + m_1(n - 1) + 4\sigma_2 + 2\gamma_2 = 2k + 2\) and \([E_1(D_4)] = [E_2(D_4)] = 2n + 2 + 2m(n - 1) + 2\beta_2 = 2k\), so \([E_1(D_3)] = [E_2(D_3)] = m + m_1(n - 1) + 4\sigma_2 + 2\gamma_2 = 2k + 2\) and \([E_1(D_4)] = [E_2(D_4)] = m + m_1(n - 1) + 4\sigma_2 + 2\gamma_2 = 2k + 2\). That is, \(D_3\) and \(D_4\) are \((4k + 2)\)-WDBC.

Similarly, we need to check if \(2n_1 + m + 3 \geq m - 1\) or \(\beta_1 < n - 2\sigma_1\), we need to check if \(2n_1 + m + 3 > m - 1\) or \(\beta_1 < n - 2\sigma_1\). Note that \(2n_1 + m + 3 = (4k - m - 2n + 4 - \alpha_1 - 2\beta_1)/(n - 1) + 3 = (4k - m - n - \alpha_1 - 2\beta_1)/(n - 1) < (mn - m - n - \alpha_1 - 2\beta_1)/(n - 1) = m - (\alpha_1 + 2\beta_1)/(n - 1)\). Besides, \(\alpha_1 + 2\beta_1 \geq n - 1\), so \(2n_1 + m + 3 \leq m - (\alpha_1 + 2\beta_1)/(n - 1)\).
and $\alpha = 4\sigma + 2\gamma$, $4\sigma + 2\gamma + 2\beta < 2n - 2$. Hence, $\beta \leq n - 2\sigma - 1 - \gamma$. If $\gamma = 0$, $\beta < n - 2\sigma - 1 < n - 2\sigma$. If $\gamma = 1$, $\beta < n - 2\sigma - 2$. In conclusion, $D_1$ and $D_2$ are well-defined cycles.

**Corollary 2.**

For integers $m$, $n \geq 4$, $T_{m,n}$ are (a) WDB bipancyclic when one of $m$, $n$ is even, the other is odd and;
(b) (2n - 4)-WDB pancyclic when $m$ is even, $n$ is odd; (2n - 4)-WDB pancyclic when $m$ is odd, $n$ is even.

**4 Conclusions**

In this paper, for one of $m$ and $n$ is even, the other is odd, we discuss whether the toroidal mesh graph $T_{m,n}$ contains a weakly dimension-balanced cycle whose length is $l$ for any integer $3 \leq l \leq mn$. We give Table 2 as a summary. Again, let $m$ be odd and $n$ be even for convenience in this table. Because the WDB pancyclicity problem on $T_{m,n}$ for both $m$, $n$ is even [9], one of $m$, $n$ is even and the other is odd (this paper) had been discussed, we want to study whether the toroidal mesh graph $T_{m,n}$ contains a weakly dimension-balanced cycle whose length is $l$ for any integer $3 \leq l \leq mn$ for both of $m$, $n$ is odd in the future.

<table>
<thead>
<tr>
<th>Table 2: Summary of this Paper</th>
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<tbody>
<tr>
<td>$m \geq 4$ is even, $n \geq 4$ is odd</td>
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<tr>
<td><strong>4k-WDBC</strong></td>
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<tr>
<td>$1 \leq k \leq \lfloor mn/4 \rfloor$</td>
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<td>[11]</td>
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<tr>
<td>Lemma 1.</td>
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<tr>
<td>$1 \leq k \leq \lfloor (mn - 2)/4 \rfloor$</td>
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<tr>
<td>Thm. 2.</td>
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<tr>
<td><strong>(2k + 1)-WDBC</strong></td>
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<tr>
<td>$n - 2 \leq k \leq \lfloor mn/2 - 1 \rfloor$</td>
</tr>
<tr>
<td>[11], Lemma 5; No, $1 &lt; k &lt; n - 2$</td>
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</table>

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**5 References**