A Discrete Three-wave System of Kahan-Hirota-Kimura Type and the QRT Mapping

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Abstract—The integrable three-wave interaction system is a well-known partial differential equation appearing in fields such as nonlinear optics and plasma physics. By eliminating the spatial derivative term from the three-wave system, we obtain the three-wave ordinary differential equation (ODE) system. Petrera et al. performed a Kahan-Hirota-Kimura discretization of the three-wave ODE system and succeeded in finding three conserved quantities of the system. However, Lax pairs for and solutions of the discrete three-wave system have not yet been obtained. In this paper, we derive three conserved quantities of the discrete three-wave system of Kahan-Hirota-Kimura type using computer algebra. Moreover, we show that in our discretized system, there is a certain variable, corresponding to the Hamiltonian of the three-wave ODE system, that is a variable in the Quispel-Roberts-Thompson (QRT) mapping and can be expressed in terms of elliptic functions. By obtaining an elliptic expression for the Hamiltonian of the three-wave ODE system, it is possible to show the stability, which means that the value corresponding to the Hamiltonian is analytic, of the difference scheme though the implicit Runge-Kutta method always possesses the A-stable.

Keywords: three-wave interaction system, Quispel-Roberts-Thompson mapping, Hamiltonian, Kahan-Hirota-Kimura discretization, Gröbner basis

1. Introduction

The study of dynamical systems has contributed significantly to developments in science and engineering. We can investigate various properties of a system, such as a natural phenomenon, by constructing a dynamical systems model that describes the behaviour of the system.

There is a special class of dynamical systems—called integrable systems—that have remarkable properties, such as conserved quantities and exact solutions, even if the systems are nonlinear.

Discrete integrable systems, derived via "integrable discretization," a process that preserves the properties of the original continuous integrable system, have been attracting attention. An understanding of these systems has led to the discovery of a close relationship between soliton equations and numerical algorithms.

In this paper, we present a discretization of an integrable three-wave system. In section 2, the three-wave interaction system is discussed. In section 3, we review the Quispel-Roberts-Thompson (QRT) mapping. In section 4, we introduce an integrable discretization called the Kahan-Hirota-Kimura discretization. In section 5, we obtain a discrete three-wave system of Kahan-Hirota-Kimura type. In section 6, we show a relationship between the discrete three-wave system of Kahan-Hirota-Kimura type and the Quispel-Roberts-Thompson (QRT) mapping. In Section 7, we conclude this paper.

2. The Three-Wave system

The three-wave system is a well-known partial differential equation (PDE) appearing in the fields of nonlinear optics and plasma physics[2]. The three-wave system is as follows:

\[
\begin{align*}
\frac{\partial z_1}{\partial t} + \alpha_1 \frac{\partial z_1}{\partial x} &= \epsilon \bar{z}_2 \bar{z}_3, \\
\frac{\partial z_2}{\partial t} + \alpha_2 \frac{\partial z_2}{\partial x} &= \epsilon \bar{z}_3 \bar{z}_1, \\
\frac{\partial z_3}{\partial t} + \alpha_3 \frac{\partial z_3}{\partial x} &= \epsilon \bar{z}_1 \bar{z}_2.
\end{align*}
\]

Here, the parameters \(\alpha_i (i = 1, 2, 3)\) and \(\epsilon\) are real numbers, and \(\bar{z}_i (i = 1, 2, 3)\) represent the complex conjugates of \(z_i (i = 1, 2, 3)\), respectively.

We consider the case where the following condition is satisfied:

\[
\frac{\partial z_i}{\partial x} = 0, \quad i = 1, 2, 3.
\]

We then obtain the following system of ordinary differential equations (ODE):

\[
\begin{align*}
\frac{d z_1}{d t} &= \bar{z}_2 \bar{z}_3, \\
\frac{d z_2}{d t} &= \bar{z}_3 \bar{z}_1, \\
\frac{d z_3}{d t} &= \bar{z}_1 \bar{z}_2.
\end{align*}
\]

Hereafter, we will refer to this system as “the three-wave ODE system.”
The three-wave ODE system can be written in the following form:

\[ H(p, q) = \prod_{k=1}^{3} p_k - \prod_{k=1}^{3} q_k \]

\[ = p_1 p_2 p_3 - q_1 q_2 q_3, \quad \text{(6)} \]

\[ \frac{dp_j}{dt} = -\frac{\partial H}{\partial p_j} = \prod_{k\neq j}^{3} q_k, \quad \text{(7)} \]

\[ \frac{dq_j}{dt} = -\frac{\partial H}{\partial q_j} = \prod_{k\neq j}^{3} p_k. \quad \text{(8)} \]

By choosing \( q_k \equiv z_k \) to be the coordinate variables and \( p_k \equiv \frac{z_k}{z_{\bar{k}}} \) to be the momentum variables, the system admits a Hamiltonian formulation and possesses a Hamiltonian \( H(p, q) \) given by (7).

3. The Quispel-Roberts-Thompson (QRT) mapping

The Quispel-Roberts-Thompson mapping (QRT mapping), introduced in [15] [16], is an 18-parameter family of birational transformations of the plane. The mapping is as follows:

\[ x^{n+1} = \frac{f_1(y^n) - x_n f_2(y^n)}{f_2(y^n) - x_n f_3(y^n)}, \quad \text{(10)} \]

\[ y^{n+1} = \frac{g_1(x^{n+1}) - y_n g_2(x^{n+1})}{g_2(x^{n+1}) - y_n g_3(x^{n+1})}, \quad \text{(11)} \]

\[ \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \quad \text{(12)} \]

\[ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} ^\top \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}, \quad \text{(13)} \]

The symbols \( \top \) and \( \times \) represent the transposition of a matrix and the outer product of two vectors, respectively.

If \( a_{jk} = a_{kj}, b_{jk} = b_{kj}, j, k = 1, 2, 3 \), the mapping is called a symmetric QRT mapping and can be written as a 3-point map.

\[ x^{n+1} = \frac{f_1(x^n) - x_{n-1} f_2(x^n)}{f_2(x^n) - x_{n-1} f_3(x^n)}. \quad \text{(14)} \]

Specifically,

\[ f_1(x) = (a_{21} x^2 + a_{22} x + a_{23})(b_{31} x^2 + b_{32} x + b_{33}) \]

\[ - (a_{31} x^2 + a_{32} x + a_{33})(b_{11} x^2 + b_{12} x + b_{13}), \quad \text{(15)} \]

\[ f_2(x) = (a_{31} x^2 + a_{32} x + a_{33})(b_{11} x^2 + b_{12} x + b_{13}) \]

\[ - (a_{11} x^2 + a_{12} x + a_{13})(b_{21} x^2 + b_{22} x + b_{23}), \quad \text{(16)} \]

\[ f_3(x) = (a_{11} x^2 + a_{12} x + a_{13})(b_{21} x^2 + b_{22} x + b_{23}) \]

\[ - (a_{21} x^2 + a_{22} x + a_{23})(b_{11} x^2 + b_{12} x + b_{13}). \quad \text{(17)} \]

Hereafter, we will refer to this mapping as the symmetric QRT mapping. Each member of this family possesses a 1-parameter family of invariant curves that fill the plane.

\[ (a_{11} + Kb_{11})(x^n)^2(x^{n+1})^2 + (a_{12} + Kb_{12})(x^n)^2(x^{n+1})^2 \]

\[ + (x^n)^2(x^{n+1})^2 + (a_{13} + Kb_{13})(x^n)^2(x^{n+1})^2 \]

\[ + 2(a_{22} + Kb_{22})(x^n)(x^{n+1}) \]

\[ + (a_{23} + Kb_{23})(x^n) + (x^{n+1}) + (a_{33} + Kb_{33}) = 0 \quad \text{(18)} \]

where the constant of integration \( K \) is invariant on each curve [16].

The biquadratic equation (18) can be parametrized in terms of elliptic functions [1]. We consider the symmetric biquadratic relations:

\[ ax^2 y^2 + b(x^2 y + x y^2) + c(x^2 + y^2) + 2d x y + e(x + y) + f = 0, \quad \text{(19)} \]

where \( x \) and \( y \) are variables (complex numbers) and \( a, b, c, d, e, f \) are given constants.

Firstly, we apply a linear fractional transformation to (19):

\[ x \to (ax + \beta)/(\gamma x + \delta), \quad y \to (ay + \beta)/(\gamma y + \delta), \quad \text{(20)} \]

where \( \alpha, \beta, \gamma, \delta \) are generally complex and \( \alpha \delta \neq \beta \gamma \). We can choose \( \alpha, \beta, \gamma, \delta \) so that \( b \) and \( e \) vanish in (19) and so that \( a = f \neq 0 \). Dividing (19) by \( a \), the biquadratic relation can be written in the following form:

\[ x^2 y^2 + 1 + c(x^2 + y^2) + 2d x y = 0. \quad \text{(21)} \]

We consider (21) to be a quadratic equation in \( y \); thus, the solution of (21) can be expressed as follows:

\[ y = -\frac{dx \pm \sqrt{-c + (d^2 - 1 - c^2)x^2 - cx^2}}{c + x^2}. \quad \text{(22)} \]

The argument of the square root is a quartic polynomial in \( x \). We can write it as a perfect square by transforming the variable \( x \) into the variable \( u \), where

\[ x = k^2 \text{ sn}(u). \quad \text{(23)} \]

Here, \( \text{sn}(u) \) is a Jacobian elliptic sn function with argument \( u \) and modulus \( k \), where

\[ k + k^{-1} = (d^2 - 1 - c^2)/c. \quad \text{(24)} \]
The argument of the square root is
\[ -c[1 - (k + k^{-1})x^2 + x^4] = -c(1 - \text{sn}^2(u))(1 - k^2 \text{sn}^2(u)) \\
= -c \text{cn}^2(u) \text{dn}^2(u). \]  
(25)

We define a parameter \( \eta \) by
\[ c = \frac{-1}{(k \text{sn}^2(\eta))}. \]  
(26)

Then from (24), we can choose the sign of \( \eta \) so that
\[ d = \frac{\text{cn}(\eta) \text{dn}(\eta)}{(k \text{sn}^2(\eta))}. \]  
(27)

Substituting these expressions into (22), it follows that
\[ y = k^2 \text{sn}(u) \text{cn}(\eta) \text{dn}(\eta) \pm \text{sn}(\eta) \text{cn}(u) \text{dn}(u). \]  
(28)

Using the addition formula, we simplify this result to
\[ y = k^2 \text{sn}(u \pm \eta). \]  
(29)

Thus, the equation for \( y \) is the same as it is for \( x \) (23) but with \( u \) replaced by \( u \pm \eta \).

The conserved quantity \( H_Q \) of the symmetric QRT mapping is as follows:
\[ H_Q = \frac{N}{D}, \]  
(30)

\[ N = a_{11}x_{n-1}^2 + a_{12}x_{n-1}x_n + a_{13}x_n^2 + a_{21}x_{n-1}^2 + a_{22}x_{n-1}x_n + a_{23}x_n^2 + a_{31}x_{n-1}^2 + a_{32}x_{n-1}x_n + a_{33}x_n^2, \]  
(31)

\[ D = b_{11}x_{n-1}^2 + b_{12}x_{n-1}x_n + b_{13}x_n^2 + b_{21}x_{n-1}^2 + b_{22}x_{n-1}x_n + b_{23}x_n^2 + b_{31}x_{n-1}^2 + b_{32}x_{n-1}x_n + b_{33}x_n^2. \]  
(32)

4. The Kahan-Hirota-Kimura discretization

In this section, we introduce a discretization method called the Kahan-Hirota-Kimura discretization. The discretization method was introduced in 1993 by W. Kahan, first appearing in his unpublished notes [11]. It is applicable to any system of ODEs for \( x : \mathbb{R} \to \mathbb{R}^n \) satisfying
\[ \frac{dx}{dt} = f(x) = Q(x) + Bx + c, \]  
(33)

where each component of \( Q : \mathbb{R}^n \to \mathbb{R}^n \) is a quadratic form, \( B \in \mathbb{R}^{n \times n} \), and \( c \in \mathbb{R}^n \). Consider a numerical integration method \( x^n \to x^{n+1} \) with a step size \( \delta \). The Kahan-Hirota-Kimura discretization reads as follows:
\[ \frac{x^{n+1} - x^n}{\delta} = Q(x^n, x^{n+1}) + \frac{1}{2} B(x^n + x^{n+1}) + c, \]  
(34)

where
\[ Q(x^n, x^{n+1}) = \frac{1}{2}(Q(x^n + x^{n+1}) - Q(x^n) - Q(x^{n+1}) \]  
(35)
is the symmetric bilinear form corresponding to the quadratic form \( Q \). It is sometimes more useful to use \( 2\delta \) for the time step size to avoid powers of 2 in the various formulas.

Kahan applied the discretization method (34) to a scalar Riccati equation and a two-dimensional Lotka-Volterra system [12]. The most remarkable feature of the scheme is that it produces solutions that stay on closed curves. Most other schemes produce solutions that either spiral in towards the equilibrium point or spiral out of the equilibrium point.

Petrera, Pfadler, and Suris applied the discretization to many integrable systems, including the three-wave system, and showed that in most cases, the discretization preserves the integrability [14].

The discretization method coincides with the following Runge-Kutta method when the applied system is restricted to quadratic vector fields [3]. Let us consider the Runge-Kutta method of order \( s \) for the following ODE (36):
\[ \frac{dx}{dt} = f(t, x). \]  
(36)

The Runge-Kutta method of order \( s \) for (36) is as follows:
\[ \frac{x^{n+1} - x^n}{\delta} = \sum_{i=1}^{s} b_i k_i, \]  
(37)

\[ k_i = f \left( \delta(n + c_i), x^n + \delta \sum_{j=1}^{s} a_{ij} k_j \right), \ i = 1, \ldots, s. \]  
(38)

It is well-known that the Runge-Kutta method can be represented by the so-called Butcher tableau, which puts the coefficients of the integrator (37) and (38) in a table as follows:
\[
\begin{array}{c|cccc}
 & A & b^T \\
\hline 
 c & 1 & b_1 & & \\
 & & & & c_1 \\
 & & & & \ldots \\
 & & & & c_s \\
\end{array}
\]  
(39)

where
\[ A = (a_{ij}), \ b = \left( \begin{array}{c} b_1 \\ \vdots \\ b_s \end{array} \right), \ c = \left( \begin{array}{c} c_1 \\ \vdots \\ c_s \end{array} \right). \]  
(40)

The Kahan-Hirota-Kimura discretization of the ODE (36) and the Butcher tableau of the scheme are expressed as follows: [3]
\[ \frac{x^{n+1} - x^n}{\delta} = -\frac{1}{2} f(x^n) + 2 f \left( \frac{x^{n+1} + x^n}{2} \right) - \frac{1}{2} f(x^{n+1}). \]  
(41)

Some properties of the discretization method follow from those of the Runge-Kutta method. For example, let us
consider the stability of a linear ODE. We introduce the stability function

\[ R(z) = \frac{\det(I-zA + zeb^T)}{\det(I-zA)}, \quad (42) \]

and the domain of absolute stability

\[ \mathcal{R} = \{ z \in \mathbb{C} : |R(z)| < 1 \} \quad (43) \]

where \( e \) stands for the vector of ones [5]. We can reduce a linear ODE to a one-dimensional linear ODE by a variable transformation of the coefficient matrix. Thus, we apply the Runge-Kutta method to the linear test problem

\[ \frac{dx}{dt} = \lambda x, \quad (44) \]

where \( \lambda \) is an eigenvalue of the coefficient matrix. The Runge-Kutta method (37) with (38) applied to the linear test problem is as follows:

\[ x^{n+1} = R(\lambda \delta)x^n. \quad (45) \]

If the domain of absolute stability \( \mathcal{R} \) contains the left half plane, the Runge-Kutta method is said to be A-stable, i.e., for any fixed \( \delta \), if the eigenvalues of the linear ODE lie in the left half-plane, the numerical method is stable. The classical Runge-Kutta method (RK4) is represented by the following Butcher tableau.

\[
\begin{array}{c|ccc}
0 & 0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} & 1 \\
1 & 0 & 0 & 1 \\
\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \\
\end{array}
\]

The stability function for RK4 is as follows:

\[ R(z) = \frac{\det(I-zA + zeb^T)}{\det(I-zA)} = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 \quad (46) \]

The region \( \mathcal{R} \) of absolute stability for the Runge-Kutta method does not include the whole left half plane. Thus, RK4 is not A-stable.

We analyse the stability of the Kahan-Hirota-Kimura discretization scheme (41). The stability function of the Kahan-Hirota-Kimura discretization is as follows:

\[ R(z) = \frac{\det(I-zA + zeb^T)}{\det(I-zA)} = \frac{z + 2}{z - 2} \quad (47) \]

The region \( \mathcal{R} \) of absolute stability of the Kahan-Hirota-Kimura discretization includes the whole left half plane \( \text{Re} z < 0 \). Thus, the Kahan-Hirota-Kimura discretization is A-stable.

5. A discrete three-wave system of Kahan-Hirota-Kimura type

In this section, we introduce a discrete three-wave system of Kahan-Hirota-Kimura type. Petrer, Pfadler, and Suris introduced a discrete three-wave system of Kahan-Hirota-Kimura type by carrying out the Kahan-Hirota-Kimura discretization of the three-wave ODE system [14].

Through the variable transformations

\[ w_1 = -\frac{z_1}{i}, \quad w_2 = -\frac{z_2}{i}, \quad w_3 = \frac{z_3}{i}, \quad (48) \]

from (5), the three-wave ODE system can be rewritten as follows:

\[ \frac{dw_1}{dt} = i w_2 w_3, \quad \frac{dw_2}{dt} = i w_3 w_1, \quad \frac{dw_3}{dt} = i w_1 w_2. \quad (49) \]

Using

\[ w_i = x_i + iy_i, \quad i = 1, 2, 3, \quad (50) \]

from (49), we obtain

\[ \frac{dx_i}{dt} = x_j y_k + y_j x_k, \quad (51) \]

\[ \frac{dy_i}{dt} = x_j x_k - y_j y_k, \quad (52) \]

where \( (i, j, k) \) represents one of the cyclic permutations of \((1, 2, 3)\). The Kahan-Hirota-Kimura discretization is as follows:

\[ \frac{x_i^{n+1} - x_i^n}{\delta} = x_j y_k^{n+1} + x_j^{n+1} y_k^n + x_j^{n+1} y_k^{n+1} + y_j^{n+1} x_k^n, \quad (53) \]

\[ \frac{y_i^{n+1} - y_i^n}{\delta} = x_j^{n+1} x_k^n + x_j^{n+1} x_k^{n+1} - y_j^{n+1} y_k^{n+1} - y_j^{n+1} y_k^{n+1}. \quad (54) \]

In matrix form:

\[ A(x, y, \delta) \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \leftrightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A^{-1}(x, y, \delta) \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad (55) \]

where

\[ A(x, y, \delta) = \begin{pmatrix} 1 & -\delta y_3 & -\delta y_2 & 0 & -\delta x_3 & -\delta x_2 \\ -\delta y_3 & 1 & -\delta y_1 & -\delta x_3 & 0 & -\delta x_1 \\ -\delta y_2 & -\delta y_1 & 1 & -\delta x_2 & -\delta x_1 & 0 \\ 0 & -\delta x_3 & -\delta x_2 & 1 & -\delta y_3 & -\delta y_2 \\ -\delta x_3 & 0 & -\delta x_1 & -\delta y_3 & 1 & -\delta y_1 \\ -\delta x_2 & -\delta x_1 & 0 & -\delta y_2 & -\delta y_1 & 1 \end{pmatrix}. \quad (56) \]

The purpose of our work is to obtain the solutions of the discrete system. Setting

\[ u_i^n = 2y_i^n - 2i x_i^n, \quad v_i^n = 2y_i^n + 2i x_i^n \quad i = 1, 2, 3, \quad (57) \]
Fig. 1: An orbit of the discrete three-wave system of Kahan-Hirota-Kimura type with initial values $(u_1^0, u_2^0, u_3^0) = (-0.36 + 0.26i, -0.28 + 0.51i, 0.52 + 0.118i)$ and $\delta = 0.1$.

from (53) and (54) we obtain
\[
(u_i^{n+1} - u_i^n) / \delta = (v_j^{n+1} v_k^n + v_j^n v_k^{n+1}) / 2, \quad (58)
\]
\[
(v_i^{n+1} - v_i^n) / \delta = (u_j^{n+1} u_k^n + u_j^n u_k^{n+1}) / 2, \quad (59)
\]

$(i, j, k)$ represents one of the cyclic permutations of $(1, 2, 3)$. Here, $v_i^n$ $(i = 1, 2, 3)$ denotes the complex conjugate of $u_i^n$ $(i = 1, 2, 3)$, respectively. Hereafter, we shall call (58)-(59) the discrete three-wave system of Kahan-Hirota-Kimura type.

6. Relationship between the discrete system and the QRT mapping

In this section, we obtain the conserved quantities of the discrete three-wave system and a relationship between the Kahan-Hirota-Kimura type discrete three-wave system and the QRT mapping.

In the three-wave ODE system (5), the quantity
\[
r = z_1 z_2 z_3 - z_1^2 z_2 z_3
\]
(60)
is a Hamiltonian and is thus conserved. However, in the discrete three-wave system (58)-(59),
\[
r^n = u_1^n u_2^n u_3^n - v_1^n v_2^n v_3^n
\]
(61)
is not a conserved quantity. From the formula for $r^n$, we can see that the Kahan-Hirota-Kimura type discrete three-wave system has a periodic solution. We assume the following biquadratic equation:
\[
a_0 (r^n)^2 (r^{n+1})^2 + a_1 (r^n)^2 r^{n+1} + a_2 r^n (r^{n+1})^2 + a_3 (r^n)^2 + a_4 (r^n)^2 + a_5 r^n r^{n+1} + a_6 r^n
\]
\[
+ a_7 r^{n+1} + a_8 = 0,
\]
(62)
where $a_i$ $(i = 0, \ldots, 8)$ are complex constants. We set the initial values $u_i^0, v_i^0$ $(i = 1, 2, 3)$, compute the time evolution $r^n (i = 0, \ldots, 9)$ through (58) ~ (59) and (61), and solve the following equations:
\[
A_{1,1} = (r^0)^2 (r^1)^2, A_{1,2} = (r^0)^2 r^1, \\
A_{1,3} = r^0 (r^1)^2, A_{1,4} = (r^0)^2, \\
A_{1,5} = (r^1)^2, A_{1,6} = r^0 r^1, \\
A_{1,7} = r^0, A_{1,8} = r^1, \\
A_{1,9} = 1, \\
A_{9,1} = (r^8)^2 (r^9)^2, A_{9,2} = (r^8)^2 r^9, \\
A_{9,3} = r^8 (r^9)^2, A_{9,4} = (r^8)^2, \\
A_{9,5} = (r^9)^2, A_{9,6} = r^8 r^9, \\
A_{9,7} = r^8, A_{9,8} = r^9, \\
A_{9,9} = 1,
\]
\[
\begin{pmatrix}
A_{1,1} & \cdots & A_{1,9} \\
\vdots & \ddots & \vdots \\
A_{9,1} & \cdots & A_{9,9}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
\vdots \\
a_8
\end{pmatrix} = 0. \quad (63)
\]
We observe that these equations have non-trivial solutions. In particular,
\[
a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = 0, \quad a_6 = a_7,
\]
(64)
Eq. (62) is as follows:
\[
(r^n)^2 (r^{n+1})^2 + a_5 r^n r^{n+1} + a_6 (r^n + r^{n+1}) + a_8 = 0.
\]
(65)
Thus, a suitable choice of $a_5, a_6, a_8$ yields the conserved quantities of the discrete system. To simplify the calculation, we assume $h_1$ and $h_2$ in the following equation are the conserved quantities:
\[
(r^n)^2 (r^{n-1} + r^{n+1}) + h_1 r^n + h_2 = 0.
\]
(66)
We obtain (66) by subtracting (65) from (67) and assuming $h_1 = a_5, h_2 = a_6$, yielding:
\[
(r^{n-1})^2 (r^n)^2 + a_5 r^{n-1} r^n + a_6 (r^{n-1} + r^n) + a_8 = 0.
\]
(67)
We obtain the following equations:

\[(r^n)^2 (r^{n-1} + r^{n+1}) + h_1 r^n + h_2 = 0, \quad (68)\]

\[(r^{n+1})^2 (r^n + r^{n+2}) + h_1 r^{n+1} + h_2 = 0, \quad (69)\]

where \(h_1\) and \(h_2\) are expressed in terms of \(r^{n-1}, r^n, r^{n+1},\) and \(r^{n+2}\). Moreover, using the computer algebra system REDUCE and the following equations:

\[
\begin{align*}
    r^{n-1} &= u_1^{n-1} u_2^n - u_1^n u_2^{n-1}, \\
    r^n &= u_1^n u_2^n - v_1^n v_2^n, \\
    r^{n+1} &= u_1^{n+1} u_2^n v_2^{n+1} + v_1^n u_2^n v_2^{n+1}, \\
    r^{n+2} &= u_1^{n+2} u_2^{n+2} v_2^{n+2}.
\end{align*}
\]

We can check that \(h_1\) and \(h_2\) are conserved quantities of the discrete three-wave system (58)+(59) by using the Gröbner basis in the computer algebra Risa/ASIR [17]. Details of the Risa/ASIR program can be found in Appendix A.2. Let \(G\) be the Gröbner basis of the discrete three-wave ODE system, and let \(h_1^G, h_2^G\) be the conserved quantities. The result is as follows:

\[
\text{Numerator } (h_1^{n+1} - h_1^n) \xrightarrow{G} 0, \quad (88)\]

\[
\text{Numerator } (h_2^{n+1} - h_2^n) \xrightarrow{G} 0. \quad (89)\]

We see that (66) can be expressed as follows:

\[
r^{n+1} = -h_1 r^n - h_2 - r^{n-1}(r^n)^2. \quad (90)\]

We note that (90) is a special case of the QRT mapping (14), where the conditions

\[
\begin{align*}
    f_1(r^n) &= -h_1 r^n - h_2, \quad (91) \\
    f_2(r^n) &= (r^n)^2, \quad (92) \\
    f_3(r^n) &= 0, \quad (93)
\end{align*}
\]

are satisfied. For this reason, the variables \(r^n = u_1^n u_2^n u_3^n - v_1^n v_2^n v_3^n\) are those in which the QRT mapping takes place. As discussed above, the integration is carried out in terms of elliptic functions. Thus, \(r^n\) is integrated in terms of elliptic functions.

The Kahan-Hirota-Kimura discretization does not strictly preserve the value of \(r\) (60). However, the discrete analogue of the Hamiltonian (61) of the Kahan-Hirota-Kimura discretization is expressed via the elliptic functions.

### 7. Conclusion

We derive the conserved quantities of the discrete three-wave system of Kahan-Hirota-Kimura type using computer algebra. Moreover, the variables \(r^n = u_1^n u_2^n u_3^n - v_1^n v_2^n v_3^n\), which correspond to the Hamiltonian of the continuous three-wave ODE system and can be expressed in terms of elliptic functions, are the variables in which the QRT mapping takes place. The QRT mapping takes place in terms of elliptic functions. Thus, \(r^n\) is integrated in terms of elliptic functions.

It is known that the solutions of the continuous three-wave ODE system can be expressed in terms of hyperelliptic functions. However, no relationship between the discrete three-wave system of Kahan-Hirota-Kimura type and elliptic functions has been found. In this paper, we show a
new relationship between the discrete three-wave system of Kahan-Hirota-Kimura type and elliptic functions via the QRT mapping.

Finding the solutions $u^n, v^n(i = 1, 2, 3)$ of the discrete three-wave system of Kahan-Hirota-Kimura type requires future work.

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References


Appendix

A Conserved quantities through computer algebra

To obtain conserved quantities through computer algebra, we use the Reduce[18] command as follows:

```plaintext
off nat on ezgcd
ans := solve({u1 - u1 = v2 * v3 + v3 * v2, uu2 - uu2 = v1 * v3 + v3 * uu1, uu3 - uu3 = v1 * v3 + v3 * uu1, uu1 - uu1 = u2 + uu3 + uu3 + uu1, uu2 - uu2 = uu1 + uu3 + uu3 + uu1}), {uu1, uu2, uu3, uu1, uu2, uu3})
```

B Checking command for conserved quantities

To check the conserved quantities, we adopt the Risa/Asir[17] command

```plaintext
h1 = (2 * (-3 * u1^6 + u2^2 * u3^2 * v1^4 + 4 * u1^6 * u2^3 * u3^2 * v1^3 * v2 + ... - 4 * u3 * v3 + 1))
```

G1 = subst(h1, u1, u1, u1, u2, uu2, uu3, v1, uu1, v1, uu2, uu2, u3, uu3) G2 = subst(h2, u1, uu1, uu2, uu2, uu3, v1, uu1, v1, uu2, uu2, v3, uu3) V = [uu1, uu2, uu3, uu1, uu2, uu3] G = nd_gr_trace([uu1 - uu1 - (v2 * v3 + v3 * uu2), uu2 - uu2 - (v1 * v3 + v3 * uu1), uu3 - uu3 - (v1 * v3 + v2 * uu1), uu1 - uu1 - (v2 * uu3 + uu3 * uu2), uu2 - uu2 - (v1 * uu3 + uu3 * uu1), uu3 - uu3 - (v1 * uu3 + uu3 * uu1), uu1 - uu1 - (v2 * uu3 + uu3 * uu2), uu2 - uu2 - (v1 * uu3 + uu3 * uu1), uu3 - uu3 - (v1 * uu3 + uu3 * uu1)], V, 1, 1, 0); load("gr")
p_nf((H1 - G1, G, V, 0); p_nf((H2 - G2), G, V, 0); end
```